Exam 3 Boot Camp: Trigonometry - Part 2

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Treat this set of exercises like an intense workout program. There are about 5 pages' worth for each day (high volume). Set time aside and plan on spreading out the work over several days, ideally factoring in some rest days. The exercises are mostly focused on one section of the book but you will find, sprinkled here and there, references to topics covered earlier in class. These may not reflect the content of what is on the exam (or even the homework and quizzes), but they are meant to test your cumulative understanding.

Contents

Day 1: Graphs of $\sin(x)$ and $\cos(x)$

Warm-up

1. Identify the graph transformations: how is the graph of $f(x)$ modified in the following functions?

   (a) $f(-x)$

   (b) $f\left(\frac{1}{2}x\right)$

   (c) $f\left(x+\frac{\pi}{6}\right)$

   (d) $f(x)-4$

   (e) $-f(x)$

   (f) $3f(x)$

2. A graph corresponds to a function if it passes the _____ _____test.

3. If the graph of $f(x)$ passes the _____ _____test, f has an inverse function.

Memory

1. A function is called *periodic* if

2. The smallest value of p (if there is one) such that $f(x+p) = f(x)$ is called the _____of $f(x)$.

3. The graphs of _____functions are symmetric about the origin. For these functions, $f(-x) =$_____.

4. The graphs of _____functions are symmetric about the y axis. For these functions, $f(-x) =$_____.

5. Let $f(x)$ be a periodic function that attains the maximum value M and minimum value m. Then the _____of $f(x)$ is equal to $\frac{1}{2}(M-m)$.

6. $\sin(x)$ and $\cos(x)$ have period _____.

7. In $a\sin(b(x-c))$ and $a\cos(b(x-c))$, a corresponds to the _____and c to the _____. The period of the graph is $p =$_____.

Computation

1. Suppose that the graph of $f(x)$ contains the point $\left(\frac{\pi}{4}, \frac{1}{2}\right)$. What point do the graphs of the following functions contain?

   (a) $f(-x)$

   (b) $f\left(\frac{1}{2}x\right)$

Advice

Apply the graph transformation to the coordinates of the given point.

(c) $f\left(x+\frac{\pi}{6}\right)$

(d) $f(x)-4$

(e) $-f(x)$

(f) $3f(x)$

2. Use the fact that $\sin(0)=0$, $\sin(\pi)=0$ and $\sin(x)$ is periodic with period _____to describe other x intercepts of the graph.

3. Use the fact that $\cos\left(\frac{\pi}{2}\right)=0$, $\cos\left(\frac{3\pi}{2}\right)=0$ and $\cos(x)$ is periodic with period _____to describe other x intercepts of the graph.

4. Describe the range of the following functions:

(a) $\sin(2x)$

(b) $-\cos(x)+1$

(c) $\sin\left(x-\frac{2\pi}{3}\right)$

(d) $4\cos(x)$

5. Find the amplitude, period, and phase shift and sketch the graph of the following functions:

   (a) $\frac{1}{3}\sin\left(x-\frac{\pi}{2}\right)$

   (b) $-10\cos\left(\frac{5}{6}x\right)$

   (c) $-\sin\left(6\left(x-\frac{\pi}{3}\right)\right)$

   (d) $5\cos\left(\frac{3}{4}\left(x+\frac{\pi}{3}\right)\right)$

6. Write an equation of the form $a\sin(b(x-c))$ with the given amplitude, period and phase shift.

   (a) Amplitude = 2, period = 3π, phase shift = $\frac{\pi}{3}$

   (b) Amplitude = $\frac{1}{2}$, period = 2, phase shift = $-\frac{\pi}{6}$

   (c) Amplitude = 1.4, period = $\frac{\pi}{2}$, phase shift = $\frac{1}{5}$

   (d) Amplitude = π, period = 2.5π, phase shift = 1

Word Problems

1. The voltage in a circuit is given by

$$V(t) = 6\cos(120\pi t)$$

   where t is the measured in seconds. The frequency is the number of cycles completed in one second. Find the amplitude, period and frequency of $V(t)$. Sketch one cycle of the function over an appropriate interval.

2. The population of a particular species of migratory birds in Florida is given by the function

$$P(t) = 3000 + 250\cos(4\pi t)$$

where t is measured in years. What is the largest number of birds present at any time? What is the smallest number of birds present at any time? How much time elapses between the occurrences of the maximum and minimum bird population?

3. Average temperatures in Tampa roughly follow a function of the form $a\cos(b(t-c)) + d$, where t represents a month (1 = January, ..., 12 = December). Use the following information to find appropriate values for a, b, c, d.

   - Temperatures repeat each year.

   - High temperatures reach a high of 91° F, while low temperatures reach 53° F.

   - The hottest month is July.

Advice
What does this say about the period?

Advice
What does this say about the amplitude?

Advice
What does this say about the phase shift?

Warning
Make sure to adjust the value of d to match the given data!

⋆ Challenge

Suppose the minimum value of a function of the form $a\sin(b(x-c))+d$, where $a>0$, occurs at $(2,10)$ and the nearest maximum value occurs at $(7,20)$. What are the amplitude and the period of the function?

Day 2: Graphs of $\tan(x)$ and reciprocal trigonometric functions

Warm-up

1. Evaluate the following limits:

   (a) $\lim\limits_{x\to 0^+} \dfrac{1}{x}$

   (b) $\lim\limits_{x\to 0^-} \dfrac{1}{x}$

   (c) $\lim\limits_{x\to \infty} \dfrac{1}{x}$

   (d) $\lim\limits_{x\to -\infty} \dfrac{1}{x}$

2. Suppose $f(-x)=f(x)$ and $g(x)=1/f(x)$. Then $g(-x)=$

3. Suppose $f(-x) = -f(x)$ and $g(x) = 1/f(x)$. Then $g(-x) =$

4. Find the slope of the line passing through points $(-1, 3)$ and $(9, 17)$.

5. Find the slope of the line passing through the point (x, y) and the origin.

6. Find a value for x between 0 and 2π such that $\sec(x) = \cos(x)$.

7. A line containing the point $(1, 2)$ and the origin makes a _____° angle with the axis.

Memory

1. The graphs of $\tan(x)$ and $\cot(x)$ have period _____.
2. The graphs of $\csc(x)$ and $\sec(x)$ have period _____.
3. In $a\tan(b(x - c))$, the period of the graph is $p =$_____.
4. $\csc(x)$ and $\cot(x)$ are undefined whenever _____= 0.

5. $\sec(x)$ and $\tan(x)$ are undefined whenever _____= 0.

6. $\sin(x) > 0$ in quadrants _____and _____, $\cos(x) > 0$ in quadrants _____and _____.

7. For values of x between 0 and π, $\sin(x)$ attains a _____(minimum/maximum) value, when $\csc(x)$ attains a _____(minimum/maximum) value.

8. Evaluate the limits:

   (a) $\lim_{x \to 0^+} \csc(x)$

   (b) $\lim_{x \to \frac{3\pi}{2}^+} \cot(x)$

   (c) $\lim_{x \to \frac{\pi}{2}^-} \sec(x)$

   (d) $\lim_{x \to 0^-} \tan(x)$

Computation

1. Use the fact that $\tan(0) = 0$ and $\tan(x)$ is periodic with period _____to describe other x intercepts of the graph.

2. Sketch the graph of the following functions over two periods:

   (a) $\tan\left(x - \frac{\pi}{4}\right)$

   (b) $-\cot\left(\frac{2\pi}{3}x\right)$

   (c) $\frac{1}{2}\sec\left(2\left(x + \frac{\pi}{6}\right)\right)$

   (d) $5\csc\left(\frac{1}{4}\left(x + \frac{\pi}{3}\right)\right)$

3. Find the range of the functions:

(a) $2\tan(x)$

(b) $-3\sec(x)$

(c) $\csc(x-1)$

4. Find the equation of a function of the form $a\tan(b(x-c))+d$ that satisfies the given conditions:

   (a) Period = π, phase shift = $\frac{\pi}{4}$, contains the point $\left(\frac{\pi}{2}, 1\right)$.

   (b) Period = 1, phase shift = $\frac{1}{2}$, contains the points $\left(\frac{1}{2}, 0\right)$ and $\left(\frac{3}{4}, 2\right)$.

   (c) Period = 2π, phase shift = π, contains the points $\left(\frac{3\pi}{2}, -1\right)$ and $\left(\frac{\pi}{2}, 1\right)$.

5. Find values for a, b, c, d such that

   (a) $\csc(bx)$ has period 3π.

   (b) $a\sec(x)$ contains the point $(0, 5)$.

(c) $\cot(x - c)$ has a vertical asymptote at $x = \frac{\pi}{4}$.

(d) $\tan(x) + d$ contains the point $\left(\frac{5\pi}{4}, -4\right)$.

6. Show that if $f(x)$ is odd, so is $g(x) = \frac{1}{f(x)}$.

Technique

Show that $g(-x) = -g(x)$ by using the fact that $f(-x) = -f(x)$.

7. Show that if $f(x)$ is even, so is $g(x) = \frac{1}{f(x)}$.

Technique

Show that $g(-x) = g(x)$ by using the fact that $f(-x) = f(x)$.

8. Show that if $f(x)$ is periodic with period p, so is $g(x) = \frac{1}{f(x)}$.

Technique

Show that $g(x + p) = g(x)$ by using the fact that $f(x + p) = f(x)$.

⋆ Determine whether the following functions are even, odd or neither:

(a) $f(x) = \cos(x)\sin(x)$

Advice

Evaluate $f(-x)$ and compare to $-f(x)$ and $f(x)$, check your answer with an online graphing tool like geogebra.org/graphing.

(b) $f(x) = \tan^2(x)$

(c) $f(x) = \cot(x)\csc(x)$

Word Problems

1. Is it possible to find a, b, c, d so that the graph of $a\sin(b(x-c)) + d$ is exactly like the graph of $-2\cos(x) + 1$? Explain.

2. A regular n-gon is a polygon with n sides that are all the same length (e.g. for $n = 3$ an equilateral triangle, for $n = 4$ a square). We say that a polygon is *circumscribed* if its sides are tangent to a circle. A circumscribed regular n-gon is drawn on the xy plane so that one of its sides lies on the line $x = 3$ with endpoints $\left(3, 3\tan\left(\frac{\pi}{6}\right)\right)$ and $\left(3, -3\tan\left(\frac{\pi}{6}\right)\right)$. Make a sketch of this situation. What is the measure of the central angle containing $\left(3, 3\tan\left(\frac{\pi}{6}\right)\right)$ and $(3, 0)$? Use the measure of this angle to find the value of n.

⋆ Challenge

Suppose $f(x)$ is an odd function and $g(x)$ is even. Determine whether the following functions are even, odd or neither. If neither, find counterexamples.

1. $\frac{f(x)}{g(x)}$

2. $f(x) \cdot g(x)$

3. $(f \circ g)(x)$

Day 3: Inverse Trigonometric Functions

Warm-up

1. A function is one-to-one if it passes the _____ _____test.
2. When using the set of all real numbers as the domain, $\sin(x)$ and $\cos(x)$ _____(are/aren't) one-to-one.
3. If f and g are inverse functions, then $f(g(x)) = g(f(x)) =$ _____.

4. Write an equivalent equation that solves for x:

   (a) $\sin^{-1}(x) = \frac{\pi}{2}$

   (b) $\cos(x) = 0$

   (c) $\tan^{-1}(x) = \frac{\pi}{6}$

   (d) $\sec(x) = -2$

5. Which of the following expressions are undefined?

   (a) $\sin^{-1}(-3)$

   (b) $\cos^{-1}(0.4)$

   (c) $\tan^{-1}(-8)$

   (d) $\csc^{-1}\left(\frac{1}{3}\right)$

Memory

1. $\sin^{-1}(x)$, $\csc^{-1}(x)$ and $\tan^{-1}(x)$ output angles x in quadrants _____and _____.
2. $\cos^{-1}(x)$, $\sec^{-1}(x)$ and $\cot^{-1}(x)$ output angles x in quadrants _____and _____.
3. For any x, $\sin(\sin^{-1}(x)) =$ _____.
4. Is it true that for every real number x, $\sin^{-1}(\sin(x)) = x$? Explain.

5. Is it true that for every real number x, $\sec^{-1}(x) = \frac{1}{\cos^{-1}(x)}$? Explain.

6. Find two different angles θ such that $\tan(\theta) = -1$. Why is it not possible for both to appear in the range of the inverse tangent function? Which one will be output by $\tan^{-1}(-1)$?

Computation

1. Find the exact value of each expression or state why it is undefined.

   (a) $\tan^{-1}(0)$

   (b) $\sin^{-1}(\sqrt{2})$

   (c) $\sec^{-1}(-2)$

   (d) $\cot^{-1}(\sqrt{3})$

   (e) $\cos^{-1}(1)$

   (f) $\csc^{-1}(0)$

(g) $\sin\left(\sin^{-1}\left(\frac{1}{5}\right)\right)$

(h) $\sec(\sec^{-1}(\sqrt{3}))$

(i) $\cot^{-1}(\cot\left(\frac{\pi}{6}\right))$

(j) $\tan(\tan^{-1}(23.3))$

(k) $\csc^{-1}(\csc(\pi))$

$\star$ $\cos^{-1}(\cos(1))$

2. Use the fact that $\sin^{-1}(-x) = -\sin^{-1}(x)$ and $\cos^{-1}(-x) = \pi - \cos^{-1}(x)$ to evaluate the following expressions:

(a) $\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right)$

(b) $\sin^{-1}\left(-\frac{1}{2}\right)$

(c) $\tan\left(\frac{\pi}{6} - \cos^{-1}\left(-\frac{\sqrt{3}}{2}\right)\right)$

(d) $\sin\left(\frac{\pi}{3} - \sin^{-1}\left(-\frac{1}{2}\right)\right)$

3. Use a sketch to find the exact value of the following expressions:

(a) $\sin\left(\cos^{-1}\left(\frac{3}{5}\right)\right)$

(b) $\cos\left(\tan^{-1}\left(-\frac{1}{4}\right)\right)$

(c) $\tan\left(\sec^{-1}\sqrt{3}\right)$

(d) $\csc\left(\tan^{-1}(4)\right)$

Word Problems

1. A ladder is propped up against a window on the side of an apartment building, reaching a height of 20ft. The base of the ladder is 45ft away from the wall. Set up an equation to find the angle θ formed by the ground and the ladder and solve for θ. Use your calculator to evaluate the measure of the angle, in degrees, to the nearest tenth.

2. A student states that if $\sin(x) = y$, then $\sin^{-1}(y)$ is coterminal with x. Do you agree or disagree? Why?

⋆ Challenge

Draw a sketch of a right triangle to represent the equation $\sin^{-1}(x) = \theta$. Write an expression for the length of the third side of the triangle. Draw another sketch to represent the equation $\cos^{-1}(x) = \varphi$[1] and write an expression for the length of the third side of the triangle. What can you say about the two triangles? Write an equation to represent the relationship between θ and φ.

[1]The Greek letter φ is called "phi," and corresponds to the sound made by the letter 'f' in English.