Exam 3 Boot Camp: Trigonometry - Part 3

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Treat this set of exercises like an intense workout program. There are about 10 pages' worth for each day (very high volume), feel free to break it up into more manageable sections. Set time aside and plan on spreading out the work over several days, ideally factoring in some rest days. The exercises are mostly focused on one section of the book but you will find, sprinkled here and there, references to topics covered earlier in class. These may not reflect the content of what is on the exam (or even the homework and quizzes), but they are meant to test your cumulative understanding.

Contents

Day 1: Fundamental Identities

Warm-up

1. Factor:

   (a) $x^2 - y^2$

   (b) $x^3 + y^3$

   (c) $x^2 - 2xy + y^2$

   (d) $x^4 - y^4$

2. Simplify:

   (a) $\dfrac{1}{1+x} - \dfrac{1}{1-x}$

   (b) $(2x - 3y)(2x + 3y)$

(c) $\dfrac{x^2 - 2x + 1}{x - 1}$

(d) $\dfrac{x^3 - 1}{x - 1}$

(e) $(x + y)^2 - (x - y)^2$

Memory

1. Write in terms of $\sin(x)$ and $\cos(x)$:

   (a) $\csc(x)$

   (b) $\tan(x)$

   (c) $\sec(x)$

   (d) $\dfrac{1}{\tan(x)}$

(e) $\cot(x)$

(f) $\dfrac{1}{\csc(x)}$

2. An identity is an equation that holds true for __________ values of x.

3. Draw a right triangle in the first quadrant of the unit circle corresponding to central angle θ. Label the lengths of the triangle in terms of $\sin(\theta)$ and $\cos(\theta)$ and state Pythagoras' theorem.

4. A function $f(x)$ is even if $f(-x) =$______.

5. Which of the six trigonometric functions are even?

6. A function $f(x)$ is odd if $f(-x) =$______.

7. Which of the six trigonometric functions are odd?

8. We can prove that an equation is *not* an identity by finding a ____________________: a value of x for which the equation is not true.

9. Starting with the Pythagorean identity that has $\sin^2(x)$ and $\cos^2(x)$, how can the other two be obtained?

Computation

1. Use fundamental identities to find the exact value of the following expressions:

   (a) $\csc^2\left(\frac{\pi}{4}\right) - \cot^2\left(\frac{\pi}{4}\right)$

   (b) $\cos(-x) - \cos(x)$

   Advice
   Think of symmetry.

   (c) $\sec(x)\cot(x)$ if $\sin(x) = \frac{1}{3}$

   (d) $3\sec^2(x) - 3\tan^2(x)$

   (e) $\tan(x) + \tan(-x)$

   (f) $(1 - \sec(x))(1 + \sec(x)) + \tan^2(x)$

   (g) $(\csc(x) - \cot(x))(\csc(x) + \cot(x))$

2. Determine whether the equation is an identity. If it is, state what kind of identity it is (reciprocal, Pythagorean, even/odd, quotient). If not, find a counterexample:

   (a) $\sin(x) = \cos(x)$

   (b) $\tan^2(x) + 1 = \sec^2(x)$

   (c) $\tan(x) + 1 = \sec(x)$

   (d) $1 - \sin^2(x) = \cos^2(x)$

   (e) $\cos(x) = \sqrt{1 - \sin^2(x)}$

   (f) $(\sin(x) + \cos(x))^2 = 1$

3. State the fundamental identities or algebra tool used at each step:

(a)

$$\sin(x)\tan(x)+\cos(x) = \sin(x)\frac{\sin(x)}{\cos(x)}+\cos(x)$$

$$= \frac{\sin^2(x)+\cos^2(x)}{\cos(x)}$$

$$= \frac{1}{\cos(x)}$$

$$= \sec(x)$$

(b)

$$\csc^2(x)+\sec^2(x) = \frac{1}{\sin^2(x)}+\frac{1}{\cos^2(x)}$$

$$= \frac{\cos^2(x)+\sin^2(x)}{\sin^2(x)\cos^2(x)}$$

$$= \frac{1}{\sin^2(x)\cos^2(x)}$$

$$= \csc^2(x)\sec^2(x)$$

4. Fill in the blanks to verify the identity and justify each step:

(a)

$$\frac{1}{1-\sin(x)}+\frac{1}{1+\sin(x)}=\frac{1}{1-\sin(x)}\cdot\frac{\qquad\qquad\qquad}{\qquad\qquad\qquad}+\frac{1}{1+\sin(x)}\cdot\frac{\qquad\qquad\qquad}{\qquad\qquad\qquad}$$

$$=\frac{1+\sin(x)+1-\sin(x)}{1-\sin^2(x)}$$

$$=\frac{2}{\qquad\qquad\qquad}$$

$$=2\sec^2(x)$$

(b)

$$\sin^3(x)-\cos^3(x)=(\sin(x)-\cos(x))(\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad)$$

$$=(\sin(x)-\cos(x))(1+\sin(x)\cos(x))$$

(c)

$$\frac{\sin(x)}{1+\cos(x)}=\frac{\sin(x)}{1+\cos(x)}\cdot\frac{1-\cos(x)}{1-\cos(x)}$$

$$=\frac{\qquad\qquad\qquad\qquad\qquad}{1-\cos^2(x)}$$

$$=\frac{\sin(x)}{\qquad\qquad}-\frac{\sin(x)\cos(x)}{\qquad\qquad}$$

$$=\csc(x)-\cot(x)$$

⋆

$$\frac{\sin(x)+\tan(x)}{\cos(x)+1} = \frac{\sin(x)+\tan(x)}{\cos(x)+1} \cdot \underline{\hspace{6cm}}$$

$$= \frac{(\sin(x)+\tan(x))(\cos(x)-1)}{\underline{\hspace{8cm}}}$$

$$= \frac{\underline{\hspace{8cm}}}{-\sin^2(x)}$$

$$= -\frac{\cos(x)}{\sin(x)} + \frac{1}{\cos(x)\sin(x)}$$

$$= \frac{1}{\sin(x)}\left(\hspace{3cm} + \underline{\hspace{3cm}} \right)$$

$$= \frac{1}{\sin(x)}\left(\frac{-\cos^2(x)+1}{\cos(x)}\right)$$

$$= \underline{\hspace{3cm}}$$

$$= \tan(x)$$

5. Verify the identity

   (a) $\cos^4(x) - \sin^4(x) = 2\cos^2(x) - 1$

(b) $\cot^2(x) + \tan^2(x) = \sec^2(x)\csc^2(x) - 2$

(c) $(1 + \cot^2(x))(1 + \tan^2(x)) = \dfrac{1}{\sin^2(x)\cos^2(x)}$

(d) $\dfrac{\tan(x) - \sin(x)}{\tan(x)\sin(x)} = \dfrac{\tan(x)\sin(x)}{\tan(x) + \sin(x)}$

6. Make the indicated substitution and simplify. Assume $0 < \theta < \frac{\pi}{2}$

(a) $\sqrt{x^2 - 1}$, $x = \sec\theta$

(b) $\frac{x}{\sqrt{1-x^2}}$, $x = \cos\theta$

(c) $\sqrt{x^2 - 9}$, $x = 3\sec\theta$

Word Problems

1. A student presented the following proof of the fact that $(\sin(x) - \cos(x))^3 = (\sin(x) - \cos(x))(1 + \sin(x)\cos(x))$ is an identity.

$$\begin{aligned}(\sin(x) - \cos(x))^3 &= \sin^3(x) - \cos^3(x) \\ &= (\sin(x) - \cos(x))(\sin^2(x) + \sin(x)\cos(x) + \cos^2(x)) \\ &= (\sin(x) - \cos(x))(1 + \sin(x)\cos(x))\end{aligned}$$

Do you agree or disagree with the student? Explain.

2. The function $\frac{2}{x^2+4}$ attains a maximum value when $x =$ ______.

Technique
A quotient with
numerator
when the der
is at its

Otherwise,

$$\lim_{x\to\pm\infty} \frac{2}{x^2+4} =$$

If $|x| \leq 1$, then

$$-\frac{2}{x^2+4} \leq \frac{2x}{x^2+4} \leq \frac{2}{x^2+4}$$

Explain why $\sec\theta$ cannot be equal to

$$\frac{2x}{x^2+4}$$

for any real number x.

Advice

Think of the domain and range of both functions.

⋆ If x and y are any two real numbers, explain why $\sec\theta$ cannot be equal to

$$\frac{xy}{x^2+y^2}$$

⋆ Challenge

Find values of t for which it is possible to have

$$\sin\theta = \frac{1-t^2}{1+t^2}$$

Advice

Set up a co

inequality of

$a \leq b \leq c$ and t

a

indep

Day 2: Angle Sum and Difference Formulas

Warm-up

1. Choose values of α and β from the set of numbers $\{0, 15, 30, 45, 60, 90, 120, 135, 150, 180\}$ so that the equation holds.

   (a) $\alpha + \beta = 75$

   (b) $\alpha - \beta = 15$

   (c) $\alpha + \beta = 105$

   (d) $\alpha - \beta = 105$

2. Choose values of α and β from the set of numbers $\{0, \frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}, \frac{\pi}{2}, \frac{2\pi}{3}, \frac{3\pi}{4}, \frac{5\pi}{6}, \pi\}$ so that the equation holds.

   (a) $\alpha + \beta = \frac{5\pi}{12}$

   (b) $\alpha - \beta = \frac{\pi}{12}$

(c) $\alpha + \beta = \frac{19\pi}{12}$

(d) $\alpha + \beta = \frac{11\pi}{12}$

3. Find a value for β so that α and β are complementary angles.

   (a) $\alpha = 34°$

   (b) $\alpha = \frac{3\pi}{8}$

   (c) $\alpha = 59°$

   (d) $\alpha = \frac{5\pi}{12}$

Memory

1. Finish stating the identities:

   (a) $\sin(\alpha - \beta) =$

   (b) $\cos(\alpha - \beta) =$

   (c) $\tan(\alpha - \beta) =$

2. Find the exact value of the expressions, if they are defined:

(a) $\cos 13^\circ \cos 77^\circ - \sin 13^\circ \sin 77^\circ$

(b) $\sin\left(\frac{\pi}{3}\right)\cos\left(\frac{\pi}{6}\right) + \sin\left(\frac{\pi}{6}\right)\cos\left(\frac{\pi}{3}\right)$

(c) $\dfrac{\tan 29^\circ + \tan 61^\circ}{1 - \tan 29^\circ \tan 61^\circ}$

(d) $\cos\left(\frac{\pi}{4}\right)\cos\left(\frac{\pi}{3}\right) + \sin\left(\frac{\pi}{4}\right)\sin\left(\frac{\pi}{3}\right)$

Computation

1. Find the exact value of the expression without using a calculator:

   (a) $\tan(105^\circ)$

(b) $\cos\left(\frac{11\pi}{12}\right)$

(c) $\sin\left(\frac{19\pi}{12}\right)$

(d) $\cos 285°$

(e) $\sin\left(\frac{7\pi}{12}\right)$

(f) $\csc 15°$

2. Find the exact value of the expressions given that $\tan\alpha = \frac{4}{3}$, with α in quadrant III, and $\cos\beta = -\frac{12}{13}$, with β in quadrant II.

   (a) $\sin(\alpha - \beta)$

   (b) $\sin(\alpha + \beta)$

   (c) $\cos(\alpha - \beta)$

   (d) $\cos(\alpha + \beta)$

   (e) $\tan(\alpha - \beta)$

(f) $\tan(\alpha + \beta)$

3. Verify the identities

(a) $\sin(90^\circ - \theta) = \cos\theta$

(b) $\cos(90^\circ - \theta) = \sin\theta$

(c) $\tan\left(\frac{\pi}{2} - \theta\right) = \cot\theta$

(d) $\dfrac{\sin(\alpha+\beta)}{\cos\alpha\cos\beta} = \tan\alpha + \tan\beta$

(e) $\dfrac{\cos(\alpha+\beta)}{\cos\alpha\cos\beta} = 1 - \tan\alpha\tan\beta$

4. Find the exact value of each expression:

   (a) $\sin\left[\tan^{-1}\left(-\frac{3}{4}\right) + \cos^{-1}\left(\frac{4}{5}\right)\right]$

(b) $\cos\left[\sin^{-1}\left(-\frac{3}{5}\right)+\cos^{-1}\left(\frac{3}{5}\right)\right]$

(c) $\tan\left[\cos^{-1}\left(\frac{4}{5}\right)+\tan^{-1}\left(\frac{2}{3}\right)\right]$

$\star$ Verify the identities:

(a) $\sin(\alpha+\beta)\sin(\alpha-\beta)=\sin^2\alpha-\sin^2\beta$

(b) $\cos(\alpha+\beta)\cos(\alpha-\beta)=\cos^2\alpha-\sin^2\beta$

(c) $\sin(\alpha+\beta)\cos\beta - \cos(\alpha+\beta)\sin\beta = \sin\alpha$

Word Problems

1. Check with a sketch that $\sin(\pi + x) = -\sin(x)$. Verify the identity using the sum formula.

2. Explain why it is possible to view the graph of $\cos(x)$ as a horizontal shift of the graph of $\sin(x)$.

3. Explain why we cannot use the formula for $\tan(\alpha - \beta)$ to prove the identity

$$\tan\left(\frac{\pi}{2} - x\right) = \cot(x)$$

⋆ **Challenge**

Verify the identities:

1.

$$\frac{\sin(\alpha - \beta)}{\sin\alpha \sin\beta} + \frac{\sin(\beta - \gamma)}{\sin\beta \sin\gamma} + \frac{\sin(\gamma - \alpha)}{\sin\gamma \sin\alpha} = 0$$

2.

$$\frac{\sin(\alpha - \beta)}{\cos\alpha \cos\beta} + \frac{\sin(\beta - \gamma)}{\cos\beta \cos\gamma} + \frac{\sin(\gamma - \alpha)}{\cos\gamma \cos\alpha} = 0$$