Limits

Make sure to read the prompts carefully and answer each question in full to the best of your ability. Make a note of where you are stuck and why so you can prepare questions. The right margin is left intentionally wide so you can add notes as you work along.

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1 Graphically

Remember

1. Use the textbook or the video lectures to write down the *precise* definition of a limit. What do we mean when we write

   (a) $\lim\limits_{x\to c^-} f(x) = L$?

   (b) $\lim\limits_{x\to c^+} f(x) = L$?

   (c) $\lim\limits_{x\to c} f(x) = L$?

Warning
Make sure to include any requirements for c and L.

2. Explain the definition *in your own words*, as you would if you had to explain it to a friend.

3. When does the limit $\lim\limits_{x\to c} f(x)$ fail to exist? List three different ways in which this can happen.

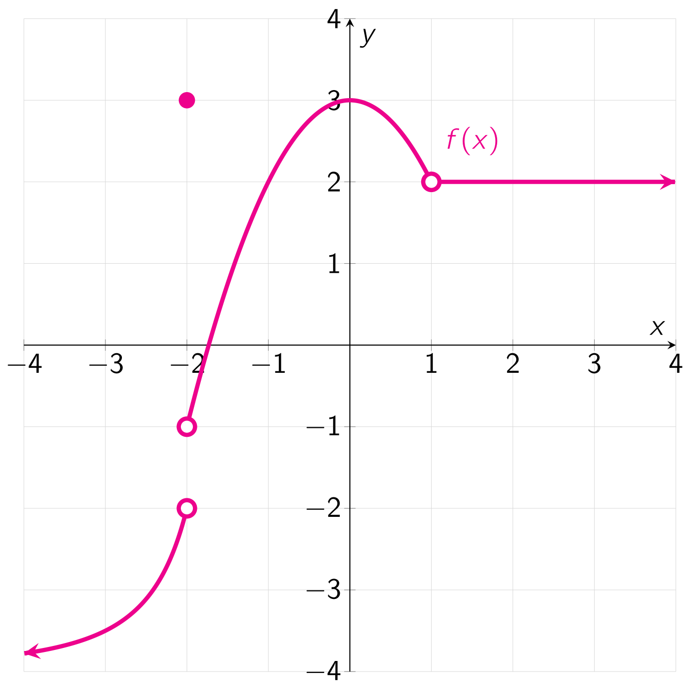
4. What is the difference between an expression that is undefined and an expression that does not exist?

Hint
A unicorn is well defined (most people will understand it to mean it's a horse with a horn), but it does not exist.

Sketch

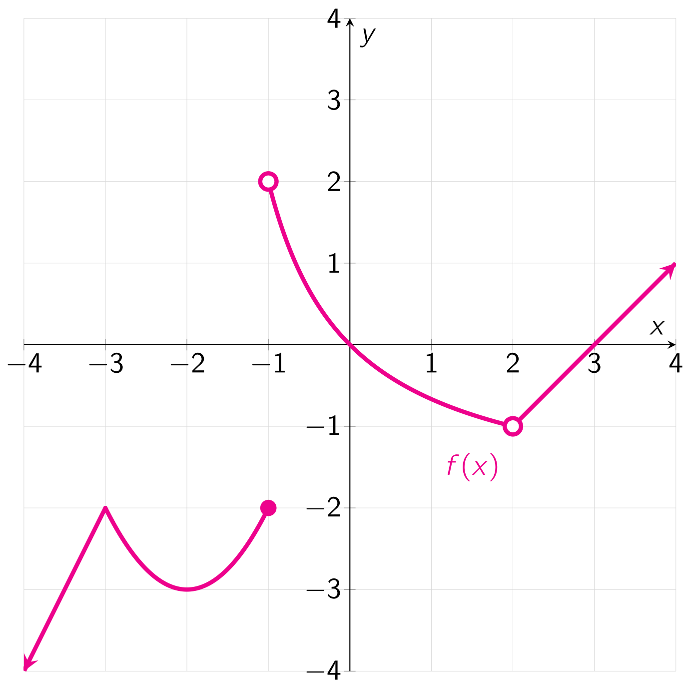
1. Use the graph below to find the following quantities. Make sure to label quantities that are undefined ("UND") or do not exist ("DNE") appropriately.

   (a) i. $\lim_{x\to -2^-} f(x) =$
   ii. $\lim_{x\to -2^+} f(x) =$
   iii. $\lim_{x\to -2} f(x) =$
   iv. $f(-2) =$
   v. $\lim_{x\to 1^-} f(x) =$
   vi. $\lim_{x\to 1^+} f(x) =$
   vii. $\lim_{x\to 1} f(x) =$
   viii. $f(1) =$



(b)

i. $\lim_{x\to -3^-} f(x) =$	ix. $\lim_{x\to 0^-} f(x) =$
ii. $\lim_{x\to -3^+} f(x) =$	x. $\lim_{x\to 0^+} f(x) =$
iii. $\lim_{x\to -3} f(x) =$	xi. $\lim_{x\to 0} f(x) =$
iv. $f(-3) =$	xii. $f(0) =$
v. $\lim_{x\to -1^-} f(x) =$	xiii. $\lim_{x\to 2^-} f(x) =$
vi. $\lim_{x\to -1^+} f(x) =$	xiv. $\lim_{x\to 2^+} f(x) =$
vii. $\lim_{x\to -1} f(x) =$	xv. $\lim_{x\to 2} f(x) =$
viii. $f(1-) =$	xvi. $f(2) =$

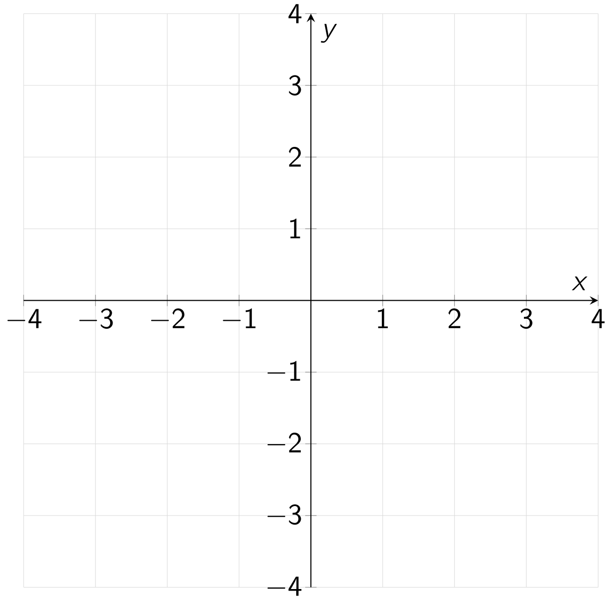


2. Draw a sketch of a function $f(x)$ satisfying the given conditions.

Warning

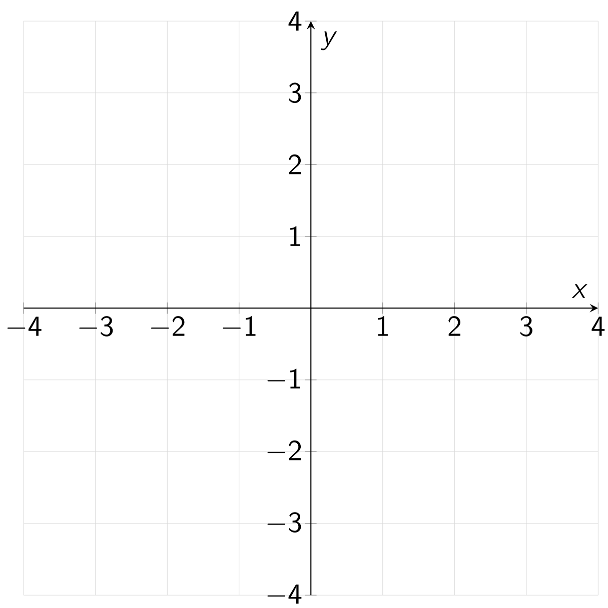
Make sure your graph passes the vertical line test.

(a) $$\lim_{x \to -3} f(x) = 2$$



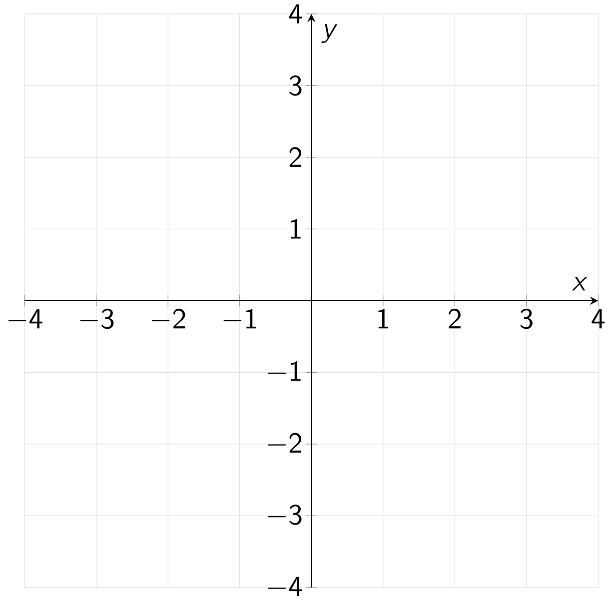
(b)

- $\lim_{x \to 2^+} f(x) = 1$, and
- $\lim_{x \to 2^-} f(x) = -3$.



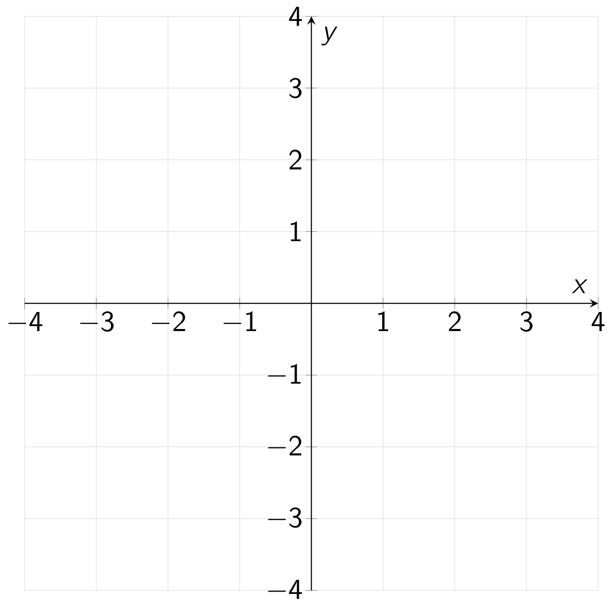
(c)

- $\lim_{x\to 0} f(x) = 0$, and
- $f(0) = 2$.



(d)

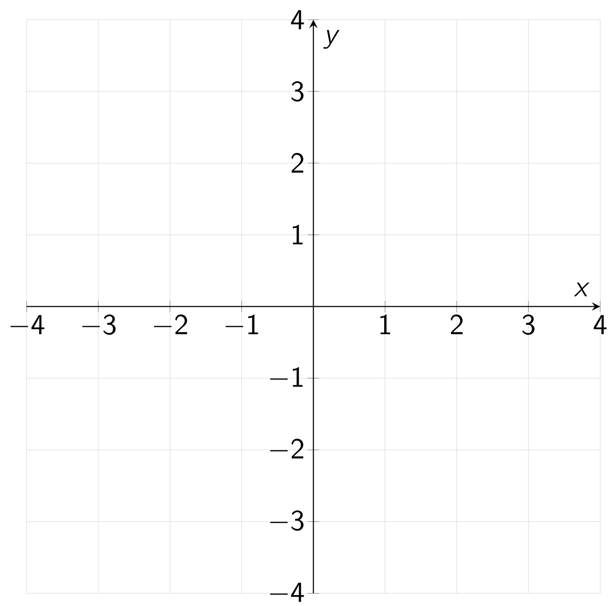
- $\lim_{x\to 1^+} f(x) = -1$,
- $f(1) = -1$, and
- $\lim_{x\to 1^-} f(x) = 4$.



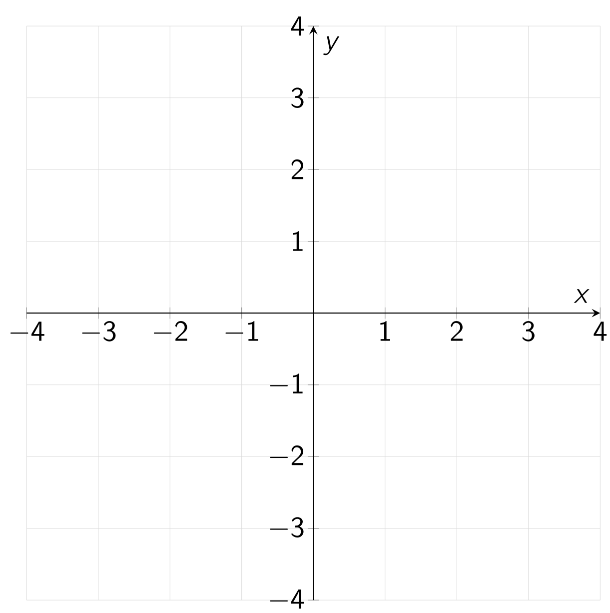
Compute

Make sketches to find the limits.

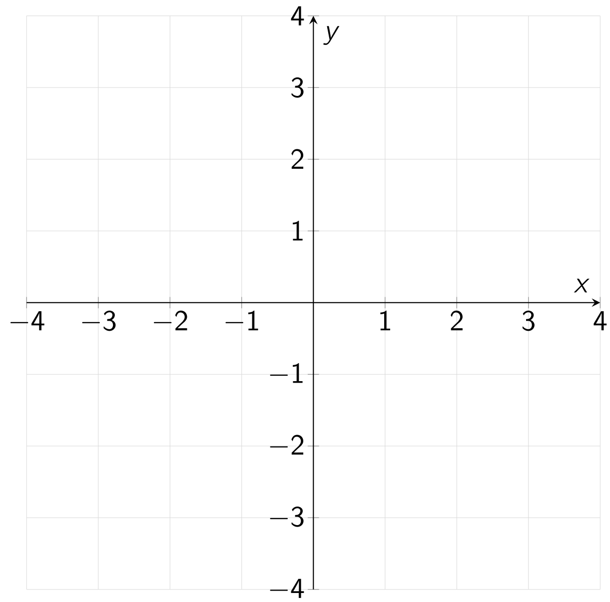
1. $\lim_{x\to -1} 1 =$



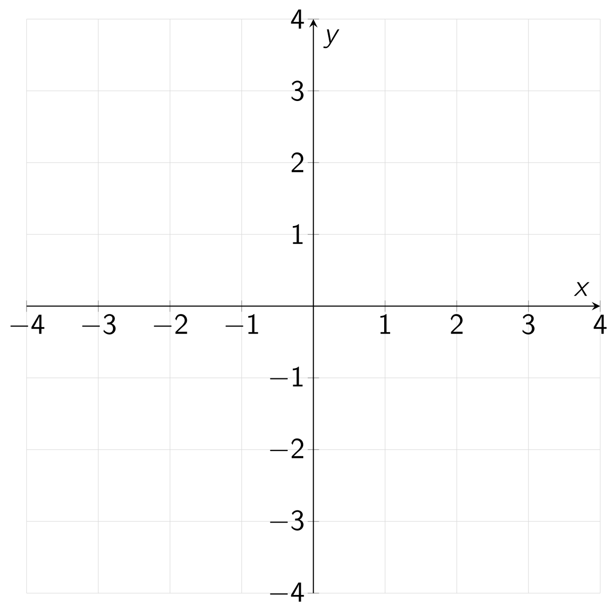
2. $\lim_{x\to 0} -2x+3 =$



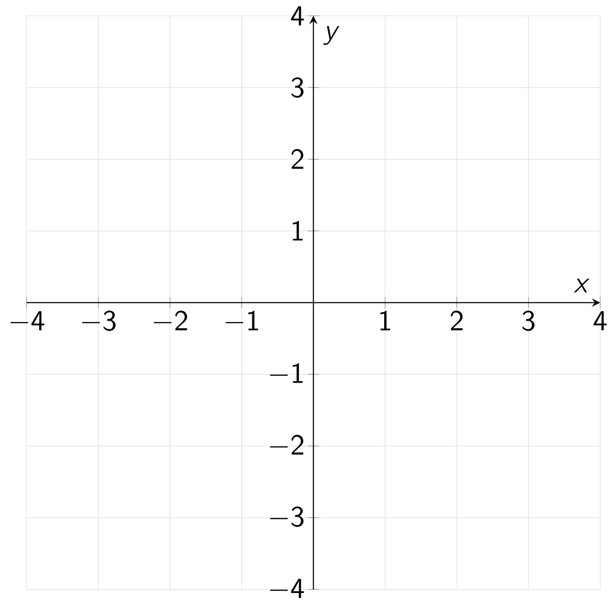
3. $\lim_{x \to 3} |x| =$



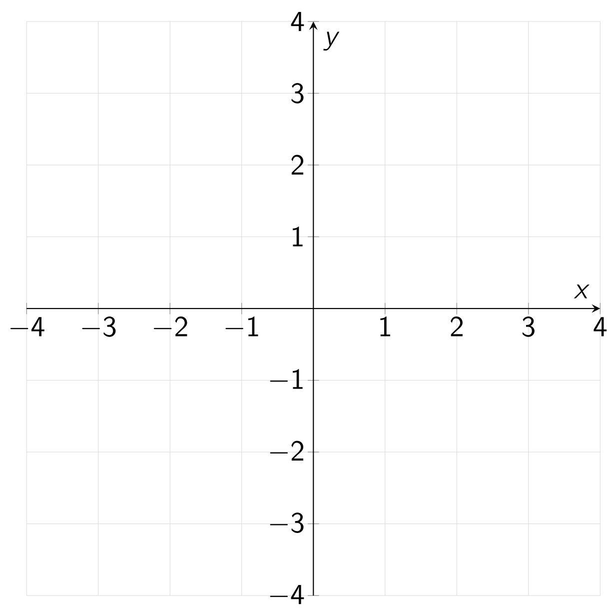
4. $\lim_{x \to -2} x^2 =$



5. $\lim_{x \to 1} \sqrt{x} =$



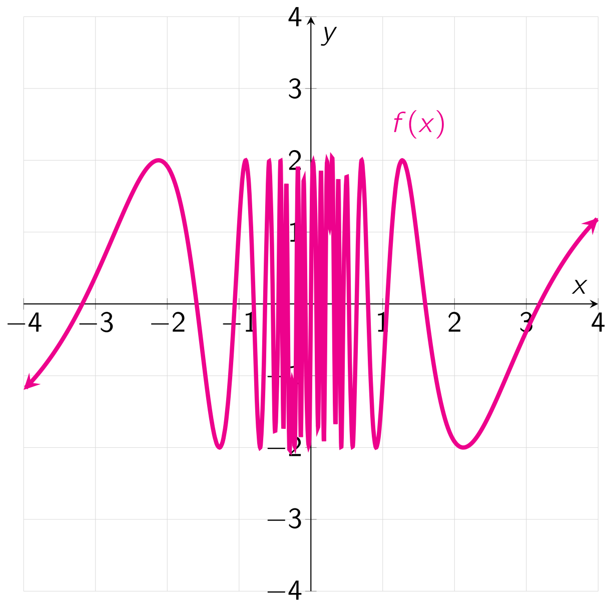
6. $\lim_{x \to 0} f(x) =$ where $f(x) = \begin{cases} 2 + x & x \leq 0 \\ -(x+1)^2 + 3 & x \geq 0 \end{cases}$



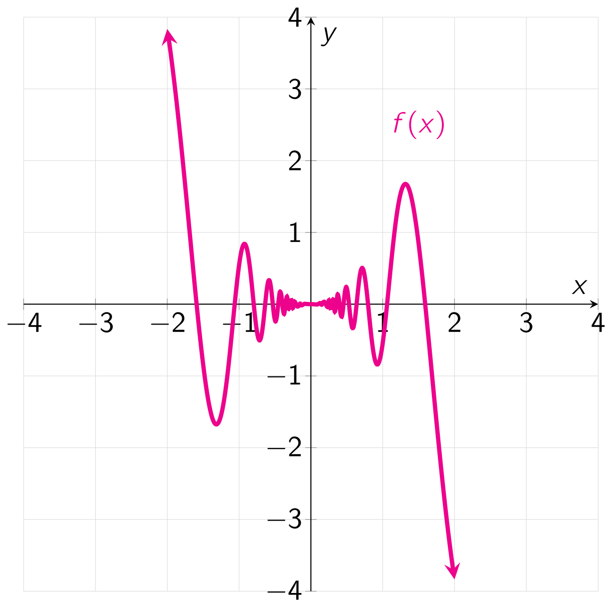
⋆ Challenge

1. Use the graph to find the limit. If the limit does not exist, explain why.

   (a) $\lim_{x\to 0} f(x) =$



   (b) $\lim_{x\to 0} f(x) =$



2. Do we need a to be in the domain of a function $f(x)$ for the limit $\lim_{x\to a} f(x)$ to exist? Give examples or counterexamples.

3. For which of the following functions might the limit as $x \to 0$ fail to exist? Explain why.

Hint
Look back at the formal definition.

- x
- $\sin(x)$
- x^2
- $\frac{1}{x}$
- $\sqrt[3]{x}$
- 2^x
- $\ln(x)$
- $\sqrt{x}$

4. Assuming the limit $\lim_{x\to a} f(x)$ exists, does it always have to equal $f(a)$? Why or why not?

2 Algebraically

Remember

Suppose c, k, A, and B are real numbers and f, g are real functions such that

$$\lim_{x\to c} f(x) = A, \quad \lim_{x\to c} g(x) = B$$

. Then

1. $\lim_{x\to c} k =$

2. $\lim_{x\to c} x =$

3. $\lim_{x\to c} kf(x) =$

4. $\lim_{x\to c} [f(x) \pm g(x)] =$

5. $\lim_{x\to c} [f(x) \cdot g(x)] =$

6. $\lim_{x\to c} \frac{f(x)}{g(x)} =$

Warning
What condition do we require of B?

7. $\lim_{x\to c} [f(x)]^k =$

8. $\lim_{x\to c} \sqrt[k]{f(x)} =$

9. A function f has the direct substitution property if for any number c in its domain

$$\lim_{x\to c} f(x) =$$

10. State the Squeeze Theorem. Make a sketch to demonstrate the result of the theorem.

Technique

When stating theorems, make sure to use "If..." and "Then..." clauses.

Compute

Use the given information to evaluate the limits. If a quantity is does not exist ("DNE"), is undefined ("UND") or there is not enough information ("NEI") make sure to say so and explain why.

Technique

Use the algebraic properties of limits.

1.
   - $\lim_{x \to -1} f(x) = 9,$
   - $\lim_{x \to -1} g(x) = 0$

   (a) $\lim_{x \to -1} (f/g)(x) =$

   (b) $\lim_{x \to -1} \frac{6}{f(x) - 3} =$

   (c) $\lim_{x \to -1} (f \cdot g)(x)$

   (d) $\lim_{x \to -1} \sqrt{f(x)}$

2.
   - $\lim_{x \to 4} f(x) = 2,$
   - $\lim_{x \to 4} g(x) = -3$

   (a) $\lim_{x \to 4} 2f(x) =$

   (b) $\lim_{x \to 4} \frac{3}{[g(x}]^2) =$

   (c) $\lim_{x \to 4} f(x) - 4g(x) =$

   (d) $\lim_{x \to 4} (f \cdot g)(x) =$

3. • $\lim_{x \to -2^-} f(x) = 0$ • $\lim_{x \to -2^+} f(x) = 0$ • $\lim_{x \to -2} g(x) = 2$

   (a) $\lim_{x \to -2^+} (f/g)(x) =$

   (b) $\lim_{x \to -2} (g/f)(x) =$

   (c) $\lim_{x \to -2^-} f(x) \cdot g(x) =$

   (d) $\lim_{x \to -2} \dfrac{g(x)}{4 + f(x)} =$

4. Use the limit laws to evaluate the following limits.

   (a) $\lim_{x \to 4} \sqrt{x + 5}$

   (b) $\lim_{x \to -2} x^2 + 3x + 6$

   (c) $\lim_{x \to 1} \dfrac{x}{\sqrt{3x + 1} - 1} =$

   (d) $f(x) = \begin{cases} x + 2 & x \leq 1 \\ 4 - x^2 & x > 1 \end{cases}$

Warning
Do not use direct substitution just yet!

5. Functions that can be obtained by combining the limit laws have the direct substitution property. For each of the following, justify your use of the direct substitution property to evaluate the limit.

   (a) $\lim_{x \to -2} \dfrac{x^3 + 2x^2 - 1}{5 - 3x} =$

   (b) $\lim_{x \to 0} \sqrt{x^2 + 7} =$

   (c) $\lim_{x \to \frac{1}{2}} \dfrac{1}{x^4} =$

6. Rewrite the expressions by factoring, multiplying by conjugates, and simplifying wherever possible so you can use direct substitution to evaluate the limits.

   (a) $\lim_{x \to 2} \dfrac{x^2 - 4}{x - 2} =$

   (b) $\lim_{x \to 0} \dfrac{x^6 - 7x^3 + 8x^2 - 5x}{x} =$

   (c) $\lim_{t \to 0} \dfrac{\sqrt{t^2 + 9} - 3}{t^2} =$

   (d) $\lim_{x \to -1} \dfrac{x^3 + 1}{x + 1} =$

(e) $\lim_{h\to 0} \frac{(3+h)^2-9}{h} =$

(f) $\lim_{x\to 16} \frac{4-\sqrt{x}}{16x-x^2} =$

(g) $\lim_{x\to -2} \frac{x^2-x-6}{3x^2+5x-2}$

7. Find bounding functions f and h such that $f(x) \le g(x) \le h(x)$ near the given value of c.

$g(x)$	c	$f(x)$	$h(x)$
$x^2\cos(x)$	0		
$\sqrt[3]{x}\arctan x - \pi/4$	$\pi/4$		
$x\sin(x)$	0		

8. If $4x - 9 \le f(x) \le x^2 - 4x + 7$ for $x \ge 0$ find $\lim\limits_{x \to 4} f(x)$.

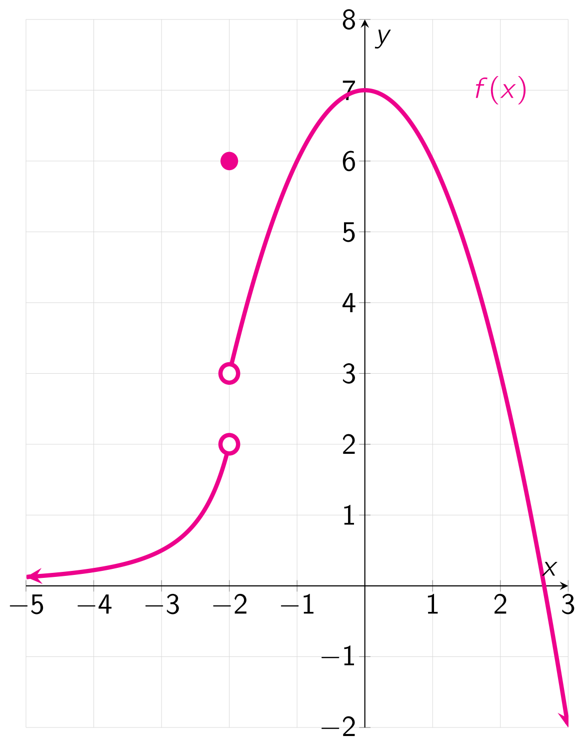
9. Use the Squeeze Theorem to show that

$$\lim_{x \to 0} x^4 \cos\left(\frac{2}{x}\right) = 0.$$

⋆ Challenge

1. Use the given information to evaluate the limits. If a quantity is does not exist ("DNE"), is undefined ("UND") or there is not enough information ("NEI") make sure to say so and explain why.

- $\lim_{x \to -2^-} g(x) = 3$
- $\lim_{x \to -2^+} g(x) = -3$
- $g(-2) = 1$
- $\lim_{x \to 0} g(x) = 5$
- $g(0) = -1$



(a) $\lim_{x \to -2} (f + g)(x) =$

(b) $\lim_{x \to -2} \dfrac{2g(x)}{f(x) - 1} =$

(c) $\lim_{x \to 0} f(x) \cdot (g(x) + 3)$

Warning

Be careful to use the right order of operations.

(d) $\lim_{x \to -2} (-5f + 2g)(x)$

2. Using the techniques used so far, evaluate the following limits.

   (a) $\lim_{x\to 3} \dfrac{x^2-9}{x^3-27} =$

   (b) $\lim_{t\to 0} \dfrac{\sqrt{1+t}-\sqrt{1-t}}{t} =$

   (c) $\lim_{x\to 3} \dfrac{\frac{1}{x}-\frac{1}{3}}{x-3} =$

3. For each of the following, either evaluate the limit or explain why it does not exist.

   (a) $\lim_{x\to 4} \dfrac{x^2+5x+4}{x-2}$

   (b) $\lim_{x^2-10x+24} x^2+12x+36$

   (c) $\lim_{t\to -5} \dfrac{2t^2+9t-5}{t^2-25}$

   (d) $\lim_{x\to -4} \dfrac{|x+4|}{2x+8}$

4. Prove that

$$\lim_{x\to 0^+} \sqrt{x}e^{\sin(\pi/x)} = 0$$

5. What is wrong with the following reasoning?

   "To find the limit as x approaches zero of $x^2 \sin(1/x)$ we choose bounding functions x^2 and $-x^2$. Therefore, to find the limit as x approaches zero of $x \sin(1/x)$ we choose bounding functions x and $-x$."

3 Infinite Limits

Remember

1. What does the symbol ∞ represent? Why is it not correct to treat it like a number?

2. Explain, in your own words, what the following expression represents

$$\lim_{x \to c} f(x) = \infty$$

3. Explain what an asymptote is in your own words.

4. Use limits to explain what we mean when we say $x = a$ is a vertical asymptote of $f(x)$.

5. As $x \to 0^+$, describe the trend in values of the following functions:

Hint
Try sketching the functions, using graph transformations where necessary.

   (a) $\dfrac{1}{x}$

   (b) $-\ln(x)$

   (c) $\csc(x)$

6. As $x \to 0^-$, describe the trend in values of the following functions:

   (a) $\dfrac{1}{x}$

   (b) $\cot(x)$

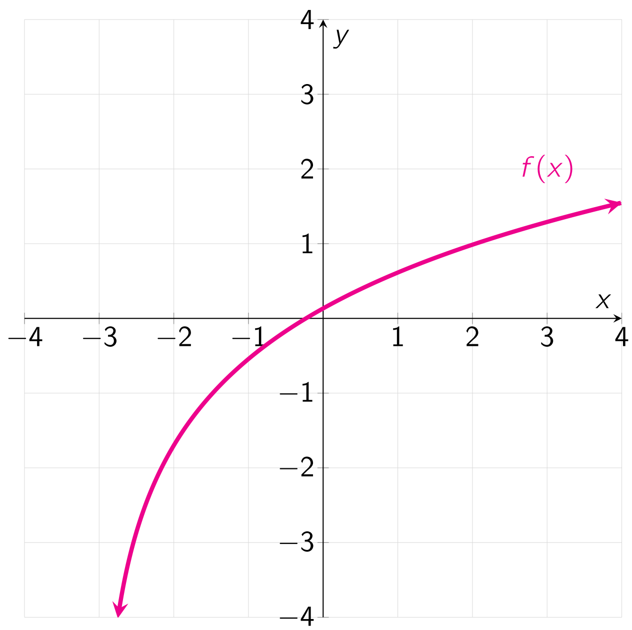
   (c) $\dfrac{1}{\ln(x+1)}$

Sketch

1. Use the graph to find the value of the limits and the equations of all vertical asymptotes. If an expression is infinite, specify whether it is $+\infty$ or $-\infty$. If an expression does not exist and is not infinite, write "DNE."

   (a) $\lim_{x \to -3^+} f(x) =$

   (b) $\lim_{x \to 2} f(x) =$



2. Use the graph to find the value of the limits and the equations of all vertical asymptotes. If an expression is infinite, specify whether it is $+\infty$ or $-\infty$. If an expression does not exist and is not infinite, write "DNE."

(a) i. $\lim_{x \to -2^-} f(x) =$

ii. $\lim_{x \to -2^+} f(x) =$

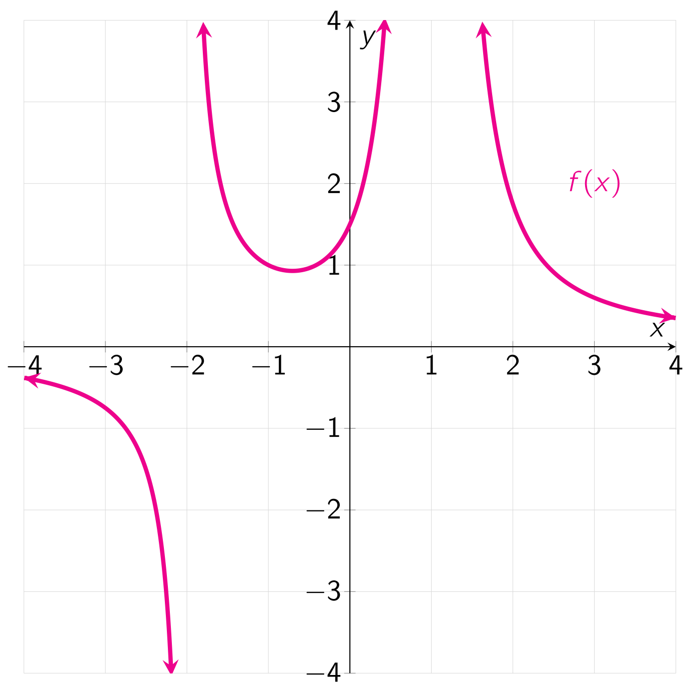
iii. $\lim_{x \to -2} f(x) =$

iv. $\lim_{x \to 1^-} f(x) =$

v. $\lim_{x \to 1^+} f(x) =$

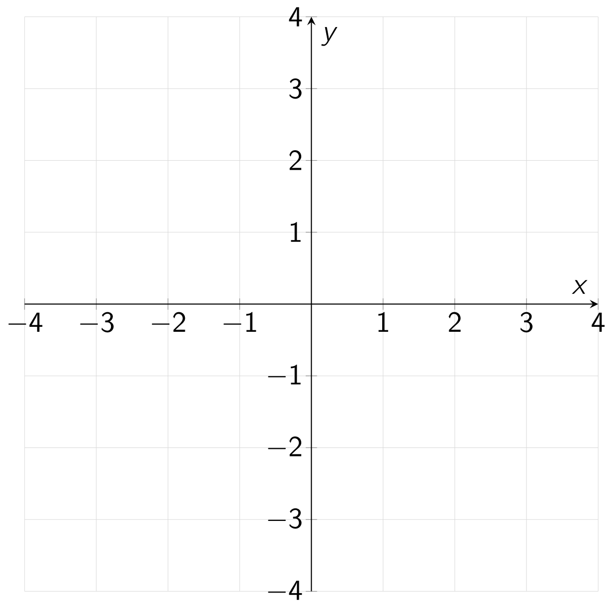
vi. $\lim_{x \to 1} f(x) =$

vii. vertical asymptotes:

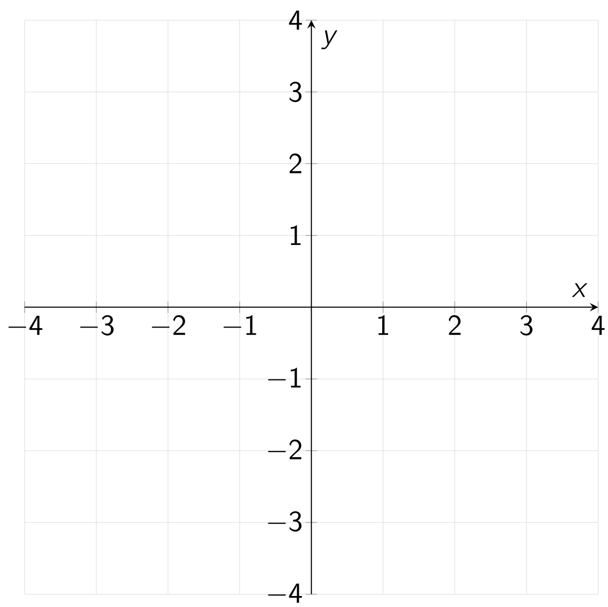


3. Draw a sketch of a function $f(x)$ that meets the given requirements.

(a)
- $\lim_{x\to -3^-} f(x) = -\infty$,
- $\lim_{x\to -3^+} f(x) = 2$,
- $f(-3) = 0$.

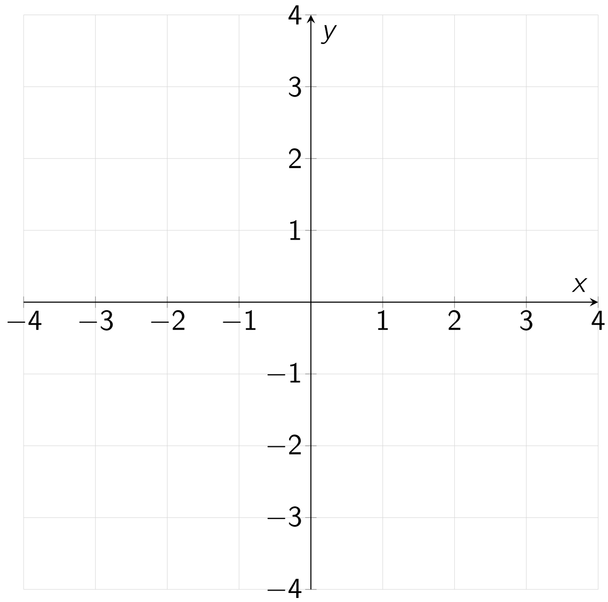


(b)
- $\lim_{x\to -1} f(x) = 0$,
- $f(-1)$ is undefined.



(c)

- $\lim_{x\to 2^+} f(x) = \infty$,
- $\lim_{x\to 2^-} f(x) = -\infty$,
- $f(2) = 0$.



Compute

1. Use the given information to evaluate the limits. If a quantity is infinite, make sure to specify whether it is $+\infty$ or $-\infty$. Limits of the form $\infty - \infty$ or $\pm\infty/\infty$ are *indeterminate*. If you identify any indeterminate form limits, mark them as "IND."

(a)

- $\lim_{x\to a} f(x) = \infty$,
- $\lim_{x\to a} g(x) = 5$

i. $\lim_{x\to a} -f(x) =$

ii. $\lim_{x\to a} \dfrac{1}{[f(x)]^3} =$

iii. $\lim_{x\to a} f(x) - 4g(x) =$

iv. $\lim_{x\to a} (f \cdot g)(x) =$

(b)
- $\lim_{x\to b^-} f(x) = \infty$
- $\lim_{x\to b^+} f(x) = \infty$
- $\lim_{x\to b} g(x) = 4$

i. $\lim_{x\to b} \frac{f(x)}{\sqrt{g(x)}} =$

ii. $\lim_{x\to b} (2g/f)(x) =$

iii. $\lim_{x\to b} f(x) \cdot g(x) =$

iv. $\lim_{x\to b} \frac{g(x)}{1 - f(x)} =$

(c)
- $\lim_{x\to c} f(x) = \infty$
- $\lim_{x\to c} g(x) = -\infty$

i. $\lim_{x\to b} (f + g)(x) =$

ii. $\lim_{x\to b} \frac{10}{f(x)} =$

iii. $\lim_{x\to b} \frac{-2}{g(x)} =$

iv. $\lim_{x\to b} g(x) - 4f(x) =$

v. $\lim_{x\to b} (g/f)(x) =$

Warning
Consider even and odd values for n separately.

2. Let n be a positive integer.

   (a) As $x \to 0^+$, what is the trend in the values of $\frac{1}{x}$?

$$\lim_{x \to 0^+} \frac{1}{x^n} =$$

   (b) As $x \to 0^-$, what is the trend in the values of $\frac{1}{x}$?

$$\lim_{x \to 0^-} \frac{1}{x^n} =$$

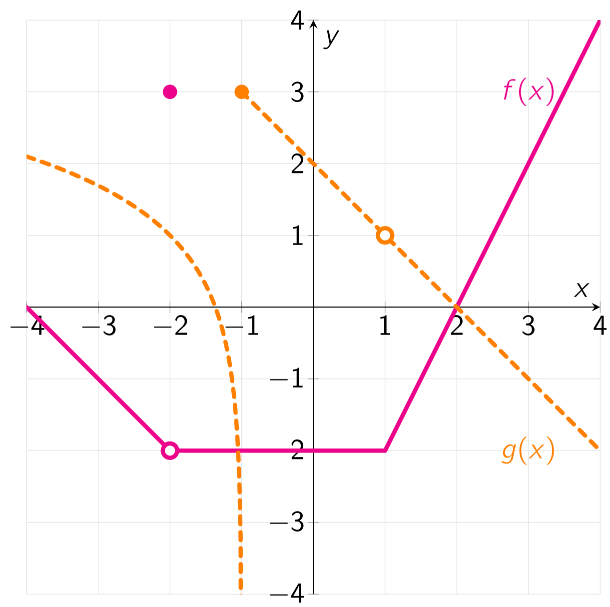
3. Use the limit laws and the graph below to evaluate the limits. Make sure to indicate if a limit is infinite, does not exist, or is undefined.

   (a) $\lim_{x \to -2^+} f(x) - g(x)$

   (b) $\lim_{x \to 0} \frac{1}{f(x) + g(x)}$

   (c) $\lim_{x \to 1} (f \cdot g)(x)$

   (d) $\lim_{x \to -1^-} g(x) \cdot [f(x) - 3]$



⋆ Challenge

1. Can the graph of a function touch its vertical asymptotes? Explain you reasoning.

2. If $x = 5$ is a vertical asymptote of $f(x)$, what do we know about the limits

$$\lim_{x\to 5^-} f(x), \quad \lim_{x\to 5^+} f(x)?$$

3. Suppose $f(x)$ has $x = 0$ as a vertical asymptote. What does this tell you about the domain and range of f?

4. Suppose $\lim_{x\to 4} f(x) = \infty$. What does this tell you about the domain and range of f?

5. Does there exist a function f with $x = -2$ as a vertical asymptote whose range is the interval $(0, 4)$? Justify fully.

6. Does there exist a function f with $x = -2$ as a vertical asymptote whose range is $(-\infty, 0) \cup (4, \infty)$? Justify fully.

4 At Infinity

Remember

1. Explain, in your own words, what the following expression represents

$$\lim_{x \to \infty} f(x) = L$$

2. Use limits to explain what we mean when we say $y = L$ is a horizontal asymptote of $f(x)$.

3. How can we use limits to find the end behavior of a function?

4. As $x \to \infty$, describe the trend in values of the following functions:

   (a) $\frac{1}{x}$

   (b) e^{-x}

   (c) $\cos(x)$

5. As $x \to -\infty$, describe the trend in values of the following functions:

   (a) $\frac{1}{x}$

   (b) $\cot(x)$

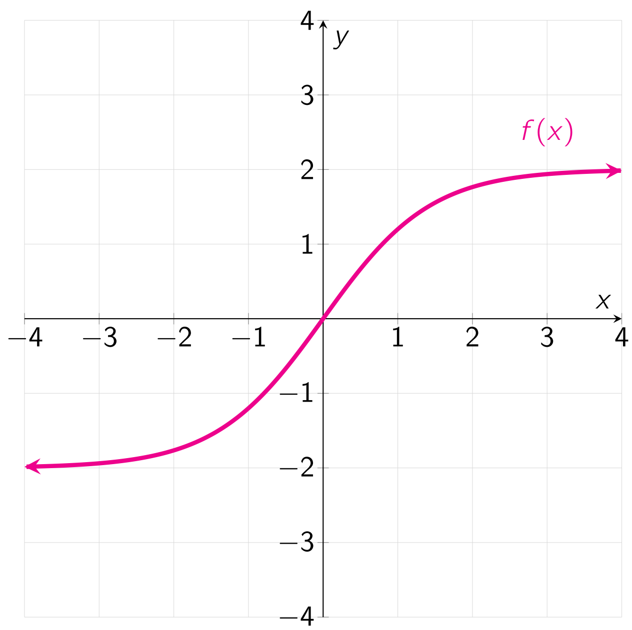
   (c) $\frac{1}{\ln(x+1)}$

Sketch

1. Use the graph below to find the following quantities. If an expression is infinite, specify whether it is $+\infty$ or $-\infty$. If an expression does not exist and is not infinite, write "DNE."

   (a) $\lim_{x \to -\infty} f(x) =$

   (b) $\lim_{x \to \infty} f(x) =$



2. Use the graph to find the value of the limits and the equations of all horizontal and vertical asymptotes. If an expression is infinite, specify whether it is $+\infty$ or $-\infty$. If an expression does not exist and is not infinite, write "DNE."

(a) i. $\lim_{x \to -\infty} f(x) =$

ii. $\lim_{x \to -1^+} f(x) =$

iii. $\lim_{x \to -1^-} f(x) =$

iv. $f(-1) =$

v. $\lim_{x \to \infty} f(x) =$

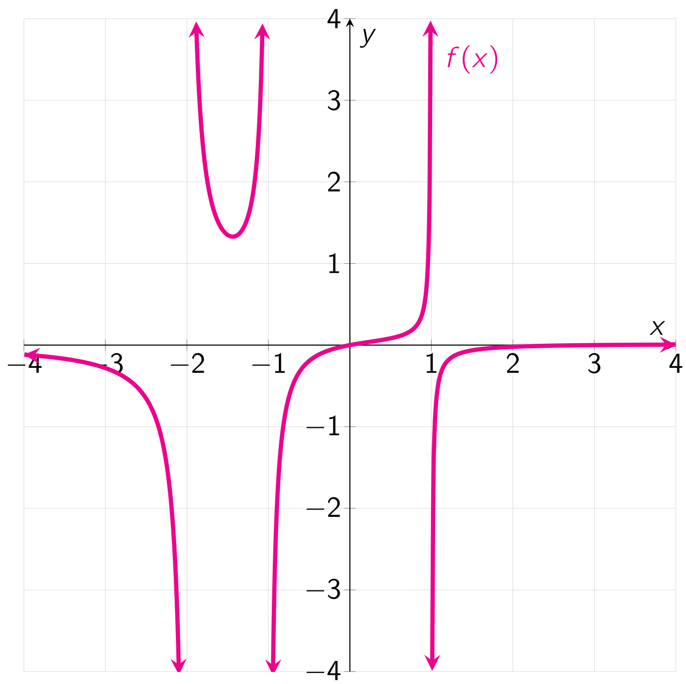
vi. $\lim_{x \to 0} f(x) =$

vii. $\lim_{x \to 1} f(x) =$

viii. $f(1) =$

ix. vertical asymptotes:

x. horizontal asymptotes:



(b) i. $\lim_{x\to-\infty} f(x) =$

ii. $\lim_{x\to-1^+} f(x) =$

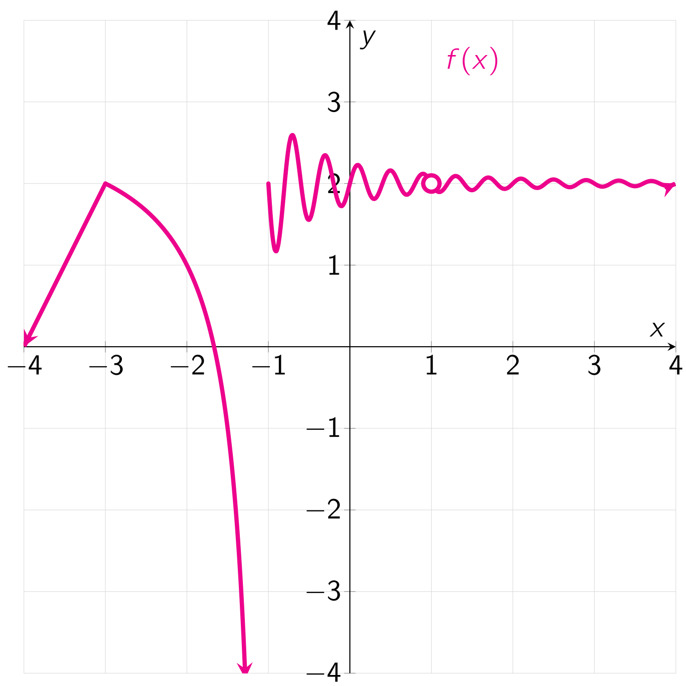
iii. $\lim_{x\to-1^-} f(x) =$

iv. $f(-1) =$

v. $\lim_{x\to\infty} f(x) =$

vi. vertical asymptotes:

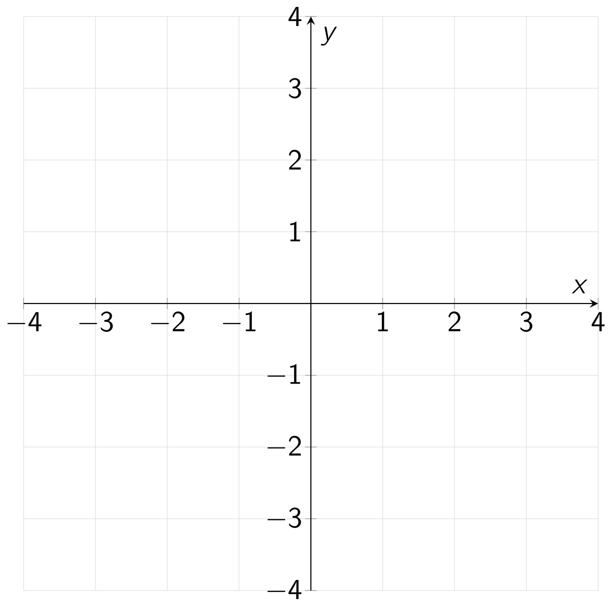
vii. horizontal asymptotes:



3. Draw a sketch of a function $f(x)$ that meets the given requirements.

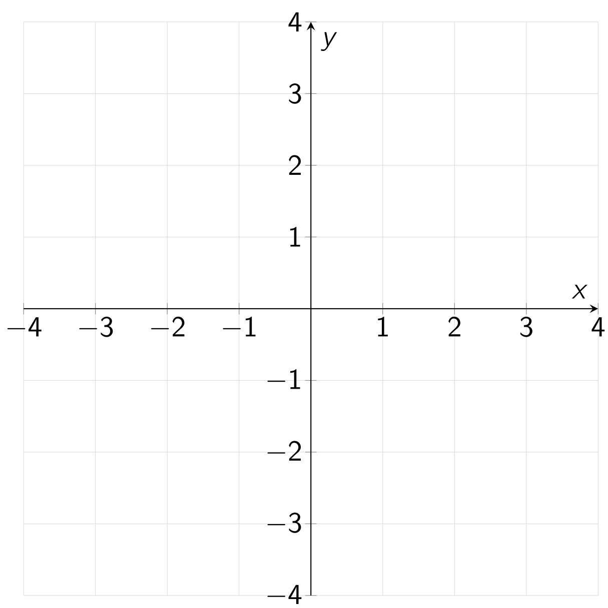
(a)

- $\lim_{x\to\infty} f(x) = 1$,
- $\lim_{x\to-\infty} f(x) = \infty$.



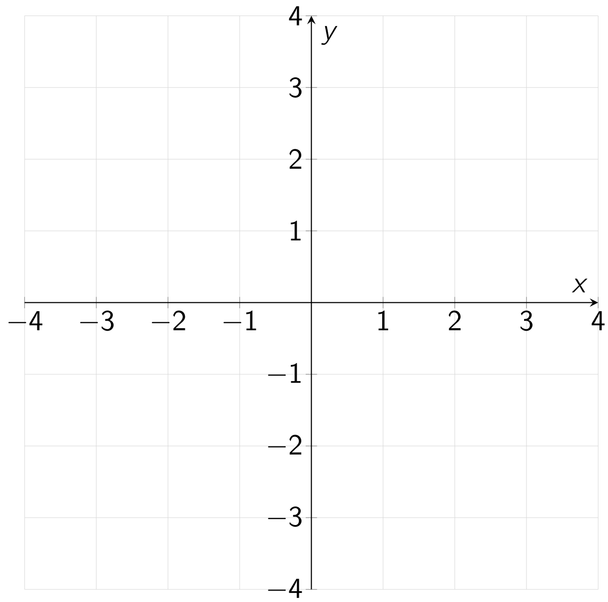
(b)

- $y = -3$ is a H.A.,
- $\lim_{x\to-1^-} f(x) = 2$,
- $\lim_{x\to-1^+} f(x) = \infty$,
- $\lim_{x\to\infty} f(x) = 1$.



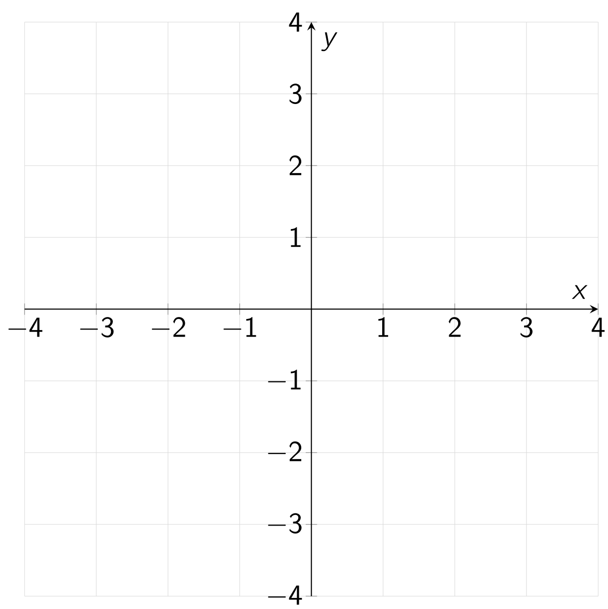
(c)

- $\lim_{x\to-\infty} f(x) = \infty$,
- $\lim_{x\to 0} = 2$,
- $f(2) = -1$,
- $f(0) = 0$,
- $x = 2$ is a V.A.,
- $\lim_{x\to\infty} f(x) = 4$.



(d)

- $f(1) = 2$,
- $x = 1$ is a V.A.,
- $\lim_{x\to\infty} f(x) = -1$.
- $\lim_{x\to 1^-} = 2$,
- $y = -3$ is a H.A.,



Compute

1. Use the given information to evaluate the limits. If a quantity is infinite, make sure to specify whether it is $+\infty$ or $-\infty$. Limits of the form $\infty - \infty$ or $\pm\infty/\infty$ are *indeterminate*. If you identify any indeterminate form limits, mark them as "IND."

   - $\lim_{x\to-\infty} f(x) = 2$
   - $\lim_{x\to-\infty} g(x) = -1$
   - $\lim_{x\to\infty} f(x) = \infty$
   - $\lim_{x\to\infty} g(x) = 8$

   (a) $\lim_{x\to-\infty} \dfrac{2}{f(x)} =$

   (b) $\lim_{x\to\infty} f(x) + g(x) =$

   (c) $\lim_{x\to-\infty} \dfrac{-6}{g(x)} =$

   (d) $\lim_{x\to\infty} \dfrac{g(x)}{f(x) + \sqrt[3]{g(x)}} =$

   (e) $\lim_{x\to-\infty} 3g(x) - \dfrac{9}{1 + f(x)} =$

Warning
Does it matter if n is even or odd?

2. Let n be a positive integer.

(a) As $x \to \infty$, what is the trend in the values of $\frac{1}{x}$?

$$\lim_{x \to \infty} \frac{1}{x^n} =$$

(b) As $x \to -\infty$, what is the trend in the values of $\frac{1}{x}$?

$$\lim_{x \to -\infty} \frac{1}{x^n} =$$

(c) Divide all terms in the quotient by x^3 and use limit laws to evaluate the limit.
$$\lim_{x \to \infty} \frac{2x^3 - 4x + 8}{x^3 - x^2 + 15}$$

(d) Divide all terms in the quotient by the highest power of x *in the denominator* and use limit laws to evaluate the limit.
$$\lim_{x \to -\infty} \frac{2x^4 - 5x^2 + 100}{x^5 + 8x^3}$$

(e) Find all horizontal asymptotes of $f(x) = \dfrac{x^2 - 8}{4x^3 - 9x^2 + 11}$.

⋆ Challenge

1. Can the graph of a function touch its horizontal asymptotes? Explain you reasoning.

2. If $y = -1$ is a horizontal asymptote of $f(x)$, can you find the limits

$$\lim_{x\to\infty} f(x), \quad \lim_{x\to-\infty} f(x)?$$

   Why or why not?

3. Divide all terms in the quotient by e^x and use limit laws to evaluate the limit.

$$\lim_{x\to\infty} \frac{e^x}{e^x - 1}$$

4. What do horizontal asymptotes tell you about the domain and range of a function?

5. Sketch functions f and g such that

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} g(x) = 2 \text{ and } \lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} g(x) = -2$$

but the range of f is $(-2, 2)$ while the range of g is $(-\infty, -2) \cup (2, 4]$.