Differential Equations
Mini Project 2: Harmonic Motion

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This project will explore some of the topics covered in class through applications in harmonic oscillators. Note that to keep the questions focused on this topic, the project may not get you to practice *all* of the learning objectives that will be assessed in the upcoming test. Look at the list in the other project and make sure you're fully prepared.

There are text fields and empty space provided for you to type in and write in your answers. You are welcome to add more pages or work in a separate document if you find it easier, but please make sure your final submission is a single pdf file that includes the rubric page at the end of this document.

1 The Quadratic Function

A second order linear ODE with constant coefficients, as you may remember, can be written as

$$ay'' + by' + c = f(x) \tag{1}$$

where a, b, c are arbitrary constants. When $f(x) = 0$, the corresponding auxiliary equation is ___________ . The differential operator associated with $ay'' + by' + c$ is ___________ . Note that both the auxiliary equation and the differential operator involve the same polynomial of degree two.

Each of the coefficients (a, b, and c) has a different effect, on both the quadratic function and on the solutions to the differential equation. To help you visualize what is happening, you can use this GeoGebra activity. If you are comfortable with coding, you are welcome to use other computer algebra software to plot the functions. While the questions here are meant to guide your analysis, you can come up with additional questions of your own to answer by experimenting with the tool. If you think it would help your explanations, you can include additional pages with plots.

1. Describe the effect that a, b and c have on the parabola by considering the following:
    - The sign of the constants
    - What happens if one or more of the constants are zero

- The magnitude of the constants (consider values $|x| < 1$, $|x| = 1$ or $|x| > 1$)
- The number of intercepts of the parabola

2. A lot of what you observed can be explained if we rewrite the quadratic function. Rewrite it in both vertex form and factored form. You can label your function anything you like and choose anything as your independent variable, but make sure to write all constants in terms of a, b and c.

3. Discuss the advantages and disadvantages of the three forms you can write your function in: expanded form (what we started out with), vertex form and factored form. What information is easiest to recover in each case? Why?

2 Second Order Equations

4. Write the general solution to Equation (1) when $f(x) = 0$. Consider all possible numbers of real roots and rewrite your solution accordingly.

5. Explain how the values of a, b and c affect the number of real roots and the resulting general solutions.

6. Show that $c_1 \cos(x) + c_2 \sin(x)$ solves Equation (1) when $f(x) = 0$ and $b = 0$.

7. Solve the homogeneous differential equation using the methods we covered in class. How does the result from Item 6 relate to your answer?

8. Discuss the effects of a, b and c on the graph of the solution to the homogeneous differential equation.

The fact that the general solution to an nth order linear differential equation can be written as a linear combination of solutions $y_1, \ldots, y_n$ is sometimes referred to as the *superposition principle*. It alludes to the fact that where the sinusoidal waves are superimposed, they can either add up or cancel out to create more complex shapes.

9. Consider $y = c_1 \sin(\omega t) + c_2 \sin(\omega t + \theta)$. Plot the two sine waves separately and their linear combination. What kind of function does y most resemble? Paste plots below and explain.

10. Use the fact that $\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$ to rewrite $\sin(\omega t + \theta)$.

11. Suppose $A\sin(\omega t + \phi) = c_1 \sin(\omega t) + c_2 \sin(\omega t + \theta)$. Substitute into the equation above and set up a system of

equations describing A in terms of c_1 and c_2. Show that $A^2 = c_1^2 + c_2^2 + 2c_1c_2\cos\theta$ and $\tan\phi = \dfrac{c_1\sin\theta}{c_1 + c_2\cos\theta}$.

12. Two sine waves don't always add up to another, bigger, sine wave, though. Consider what happens if you add two waves with different frequency. The function $y = c[\sin(\omega_1 t) + \sin(\omega_2 t)]$. Use the identity $\sin\alpha + \sin\beta = 2\sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)$ to rewrite the function.

13. Plot graphs and describe the change in behavior you observe.

14. Try creating functions

$$y = \sum_{i=1}^{n} \sin(\omega_i t)$$

for different frequencies ω_i and varying values of n ($n = 3, 5, 10$). Plot the individual curves as well as y and describe your findings. Why might this sort of function be useful to model periodic signals?

3 Harmonic Oscillators

One important application of second order linear ODEs is in the study of springs, pendulums and other oscillating objects.

Newton's second law ($F = ma$) can be used to write differential equations regarding the position of an object (if we use the fact that acceleration is the second derivative of position). In the case of an oscillator, we will assume the position function, $y(t)$, measures displacement from equilibrium[1]. In the case of a mass suspended from a spring, the system is at equilibrium when the spring is not stretched and the mass is not moving. For most springs, the force that resists the movement[2] of the mass is directly proportional to the displacement:

$$F_{\text{spring}} =$$

15. Rewrite the equation as a differential equation on the dependent variable y:

The proportionality constant k is known as the ***stiffness*** of the spring. The stiffer a spring is, the more force it takes to make a mass move from equilibrium.

[1] Here equilibrium is used in the physical sense.
[2] Read: acts in the direction opposite to movement

16. Plot a few of the solution curves for varying values of k.

17. How does stiffness translate to properties of the solution curves?

18. Do your solution curves look realistic? What do you expect the end behavior of a solution to be? What do you see in your plots?

This last equation, known as *Hooke's law* only works for very small displacements (which limits how useful it is in the real world). Something most mechanical systems have and we didn't account for is friction. Friction also resists movement, and is proportional to velocity. Write as a differential equation:

$$F_{\text{friction}} =$$

The proportionality constant here, which we will label b, is called the ***damping coefficient***. Other forces that may be acting on the oscillator are considered *external* to the system and grouped under F_{ext}. Putting everything together,

$$\begin{aligned} F &= ma \\ &= F_{spring} + F_{friction} + F_{ext} \\ &= \\ \text{Therefore } my'' &= \\ &= F_{ext} \end{aligned} \tag{2}$$

19. Look up the definition of the word "damping." Pair the coefficients in Equation (2) to those in Equation (1) and consider the effect of the damping coefficient on solution curves. Do your observations agree with the definition of the word?

20. *Unforced damped motion* is what we observe when $F_{ext} = 0$. Look up definitions of undamped, underdamped, critically damped and overdamped motion. Relate the definitions to the coefficients in Equation (1) and your observations about the parabolas in the first section.

21. Relabeling a few things in our result from Item 6, and defining $\omega = \sqrt{k/m}$, we can see that $\cos(\omega t)$ is a solution to $my'' + ky = 0$. The term ω represents the *angular frequency* of the solution. It is ___________ proportional to $\sqrt{k}$ and ___________ proportional to $\sqrt{m}$, so it ___________ when k increases and ___________ when m increases. In other words, the stiffer the spring, the ___________ the mass oscillates and the heavier the mass, the ___________ it oscillates.

4 Solutions as Responses

The homogeneous differential equation in Equation (1) assumes that there are no external forces. When this term is not zero, we can view the solutions as *responses* to the external force.

Consider

$$my'' + ky = \cos(\omega_f t) \tag{3}$$

where ω_f is a *forcing frequency*.

22. Show that the solution to Equation (3) can be written as

$$y = \frac{\cos(\omega_f t)}{m(\omega^2 - \omega_f^2)} + c_1 \cos(\omega t) + c_2 \sin(\omega t)$$

23. What happens when $\omega_f \to \omega$?

24. Find the response to a constant force and an exponential force.

In this video, Dr. Gilbert Strang explains how you can approximate the response to a discontinuous force (called an impulse here). It involves the use of the Dirac delta function and the Heaviside function.

25. Read a bit about the two impulse functions and make sketches (or paste plots) of them below.

26. Explain how you would interpret these forces physically (on a spring, or a pendulum).

27. Why would it be difficult to apply the techniques we have learned about so far to discontinuous functions?

28. Write out the steps to find an approximate response to the Dirac delta function and the Heaviside function. You can follow along with the video, but make sure to explain each step (rather than just repeating what professor Strang writes on the board).

Rubric

	1	2	3	4	5
Math concepts	Lots of work missing/incorrect/requires more advanced tools	Some work missing/incorrect/requires more advanced tools.	About half of the work is complete and correct.	Most explanations are complete and correct, using only the tools learned so far.	All explanations are complete and correct, using only the tools learned so far.
Math work	Many solutions are incorrect or incomplete	A few solutions are incorrect or incomplete	Some solutions are missing steps or have small errors	A few solutions are missing steps or have small errors	All solutions are correct and complete
Use of technology	No evidence of CAS being used.	CAS was sometimes used, but it's hard to figure out where or how.	Technology was used to aid answers, but it is not clear what commands/inputs led to the results shown.	Most questions where it is needed show appropriate use of CAS.	All questions where it is needed show appropriate use of CAS.
Clarity	It is hard to read/follow the work	Some of the work is hard to read/follow	The organization/tidiness leaves room for improvement but is readable	The work is generally easy to read/follow	It is very easy to read/follow the work done
Analysis (worth double)	Very low effort or lots of answers missing	Many low effort answers	A few answers can use work but many show a good analysis	Most explanations are in depth and show an effort to understand concepts	All explanations show an effort to master the concepts discussed