Differential Equations
Mini Project 4: (The Original) Neural Networks

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This project will explore some of the topics covered in class through the study of neural networks (from neuroscience, predecessors of the neural networks used in machine learning). Look at the list of learning objectives in the other project and make sure you're fully prepared for the upcoming test. The format of this project is a bit different from the previous ones. You can find supplementary materials in Canvas in the Mini Projects folder. Make sure your final submission is a single pdf file that includes the rubric page at the end of this document.

1 A Bit of Neuroscience

Neurons, the cells responsible for most things your brain does (helping you move, telling your heart to beat, thinking through math problems), are quite different from most other cells in your body. While most organs and tissues require some degree of specialization from the cells that make them up, and many even have complex mechanisms to communicate among themselves, none are as specialized or interconnected as neurons.

Very roughly speaking, we can simplify a neuron as a box storing a number. When the number exceeds a threshold value, the neuron fires or activates. Neural processes, such as memory recall, often[1] consist of sequential firing of neurons. Neural networks, the ones inside your brain, consist of interconnected neurons that tend to fire together in a pattern corresponding to a particular process. Seeing how artificial intelligence aims to replicate human intelligence[2], anything we can use to accurately model "real" neural networks can help us better train the artificial ones.

2 A Bit of Graph Theory

A ***graph*** in mathematics, refers to a pair of sets $G = (V, E)$. The set V contains the ***vertices*** of G, while the set E contains the ***edges***. Graphs are often depicted as collections of dots connected with lines (if the direction does not matter we call this an undirected graph), or sometimes arrows (which makes the graph a directed one).

[1] With some oversimplification

[2] This can be an area of debate, but at least *some* AI starts here

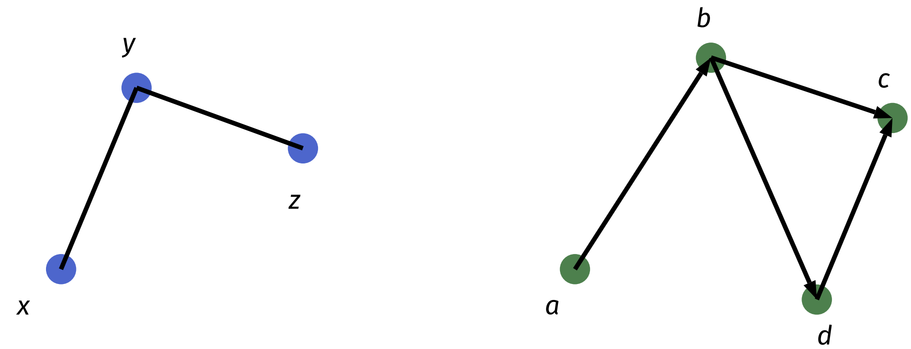

Figure 1: The undirected graph on the left has three vertices and two edges. The directed graphon the right has four vertices and four edges.

The ***degree*** of a vertex is the number of edges that connect to it. For example, vertex y in Figure 1 has degree 2. In directed graphs, we can define *in*-degree as the number of edges pointing to a vertex, and *out*-degree as the number of edges coming out of it. For example, vertex b in Figure 1 has in-degree 1 and out-degree 2.

What do graphs have to do with neuroscience? We can model neurons with vertices and use edges to indicate which ones are interconnected. Better yet, if we use directed edges we can indicate whether one neuron is activated by another. Neurons don't just fire together, but are activated sequentially as a response to some kind of stimulus when it is high enough.

3 A Bit of Parametric Curves

A ***parametric curve*** is a map $\vec{r}(t) : \mathbb{R} \to \mathbb{R}^n$. This can be visualized as a curve in n dimensional space, traversed over time as t increases. An important skill to practice when understanding parametric curves is the relationship between the components of $\vec{r}(t)$ and the behavior of the curve as a whole. For example, the parametric curve

$$\vec{r}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} \cos t \\ \sin t \end{pmatrix}$$

traces a circle on the plane, but its component functions are sinusoidal. Looking at them together, we can see that a point on the circle moves up when $y(t)$ increases and left when $x(t)$ decreases. You can see it in this Geogebra activity by moving the slider.

4 Introductory Slides

Your task for this project will be to make (some) sense of the book chapter that was posted on Canvas. The file name is *Predicting Neural Network Dynamics via Graphical Analysis*. While there are published scientific papers by Dr. Curto on the topic and this continues to be an active area of research, the book chapter may be slightly more accessible. That said, don't jump into it just yet. You may want to read through the slides she prepared here.

As you read, make note of terms and concepts that are new and those that are familiar. Not just a mental note, an actual written notes. Sort through the terms and highlight the ones that seem important to understand the gist of what the slides say, and cross out the ones that you can live without. For example, you may have a hard time moving forward without understanding what an attractor is, but if you never learn how the hippocampus is different from a hippopotamus you may still have understood a thing or two about TNLs. Your goal is not to understand *everything*, only to get the an idea of the big picture result and why it may be an important development.

The project will try to help you navigate the unfamiliar territory and make some peace with the unknown. You will be asked to record notes as you read and paste them or copy them below. These do not have to be neat, only demonstrate that you are using some kind of system to work through the material.

1. Include some notes about the familiar and unfamiliar terms below.

2. In class, we used the phase plane and direction fields to describe the space of solutions to differential equations. You'll notice the slides have pictures of surfaces with red dots on them. How do you think these are related?

3. The slides sometimes show parametric curves in $\mathbb{R}^2$ next to some of their component curves. How are the end behaviors of the two related graphically?

4. Where differential equations appear in the slides, what kind are they?

5. Though you may have heard (and used) the word in English before, it may not be intuitively clear from the pictures shown. Watch this video and summarize what chaos means in the context of dynamical systems.

5 Blinders

If you have your mathematical thinking cap on, you may have paused at the mention of "the big picture." How would you know what that picture even is? Perhaps readers from different backgrounds may want to focus on different aspects of the book chapter, so to be precise your "big picture" will be what can be seen in the slides on Canvas.

6. Repeat the exercise with the familiar and unfamiliar terms.

7. Now may be a good time to go back to the puzzles in Dr. Curto's first set of slides. Try to solve the second puzzle and explain your reasoning.

8. What appears to be the main result in the slides? Why should anyone be interested in what they were able to prove?

9. How much of what is on the second set of slides could you confidently explain to an audience of your peers? What parts are still not clear enough to you? Make a note of what you would need to learn before you felt confident trying to explain this to others.

With that in mind, *now* you may begin to read through the chapter.

6 Diving In

I strongly suggest you skim through at first and cross out any information you don't think is critical to understand the big picture.

10. Start with the appendix. Conveniently, it overlaps with the work we did to solve systems of linear differential equations. If any information is included here that is new, or you needed to review, make a note of it here.

11. Next, skip to section 8.3.1. Highlight the information that is new. You do not need to paste it here. Odds are, cliques are highlighted. They appear to be important to understand this work. Why do you think that is? Connect this back to neuroscience.

12. Now, read through the introduction. Think of this as a preview of what the authors want you to grasp. Write down three ideas you will try to focus on understanding as you make your way through the text. If you are stuck, think back on the sets of slides from earlier to narrow down your list.

13. Armed with this, skim through the rest of the text. Keeping in mind your list of three main learning objectives, feel free to skip through sections that don't seem directly relevant. Make notes as you go of what is in each section so you can come back to it if necessary. As you near on the end of the document, reflect back: can you briefly expand on the three main ideas you set out to understand?

14. It is possible (likely even) that you ran across content that both seemed important and yet unreachable. What are the questions you would need additional help answering? Can you narrow down what it is that prevents you from understanding?

Prepare a script to go along with the second set of slides if you were to present it. Don't write down what is already in the slides, think of what you would say out loud as you present. If there are slides you think are unnecessary or need to be edited, make notes of that here. If there are slides you would add, include that as well.

Rubric

	1	2	3	4	5
Math concepts	Lots of work missing/incorrect/requires more advanced tools	Some work missing/incorrect/requires more advanced tools.	About half of the work is complete and correct.	Most explanations are complete and correct, using only the tools learned so far.	All explanations are complete and correct, using only the tools learned so far.
Script (worth double)	The script doesn't add much to what is already on the slides.	The script makes a few additions but does not fully explain the content.	The slides are adequately explained by the script, but there is no reflection on whether changes are necessary.	The script explains the slides and expands on them, but some topics need work and there are no notes to suggest what is missing.	The script explains a majority of the slides and expands on them. Where topics need work, this is appropriately commented on with a plan to make improvements.
Clarity	It is hard to read/follow the work	Some of the work is hard to read/follow	The organization/tidiness leaves room for improvement but is readable	The work is generally easy to read/follow	It is very easy to read/follow the work done
Analysis (worth double)	Very low effort or lots of answers missing	Many low effort answers	A few answers can use work but many show a good analysis	Most explanations are in depth and show an effort to understand concepts	All explanations show an effort to master the concepts discussed