Engineering Calculus II
Mini Project 2: Not Quite Rocket Science

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This project will help you solve problems inspired by aeronautics. Bear in mind that with the current tools we have learned about so far, we can only set up very simplified versions of the problems we'd like to solve. One of the goals is for you to identify when approximations are necessary, when they are sufficient, and what we are missing in order to produce better solutions.

When exact solutions are possible with the techniques we have learned about in class so far, you are expected to show your work computing them by hand. You may use the table of integrals that is posted on Canvas if it will make computations easier, just remember to cite which formula you are using. For everything else, we will rely on a few different online tools.

1 Deliverables

Make sure to read the instructions carefully and show work for to answer all questions. Check that you have included

1. Handwritten solutions for the problems where this is possible/practical.
2. Citations for any formulas (like the ones in the table of integrals, but any others you find useful) you used.
3. Full sentence explanations for all the discussion questions.
4. Images of plots, screenshots, tables, or sketches where appropriate.
5. A citation to whatever tool you used to assist when evaluating difficult expressions.

You have a week to complete the project. If anything in the instructions is not clear, or need a bit of help with any of the steps, just ask. The sooner you start, the sooner you can ask for help.

2 Tools

- Wolfram Alpha. This is the free version of some of the things Mathematica can do. You can either type things like "`integrate x^2 from -2 to 3`" and it will know you mean $\int_{-2}^{3} x^2 dx$ or use the math input tool. In cases where we'd like to use formulas, it is able to understand commands like "`integrate 1/ln(x) using simpson's rule wit`" so you don't have to manually type all the terms.

  You may have run across Symbolab before, and it has many of the same (but not all) features.

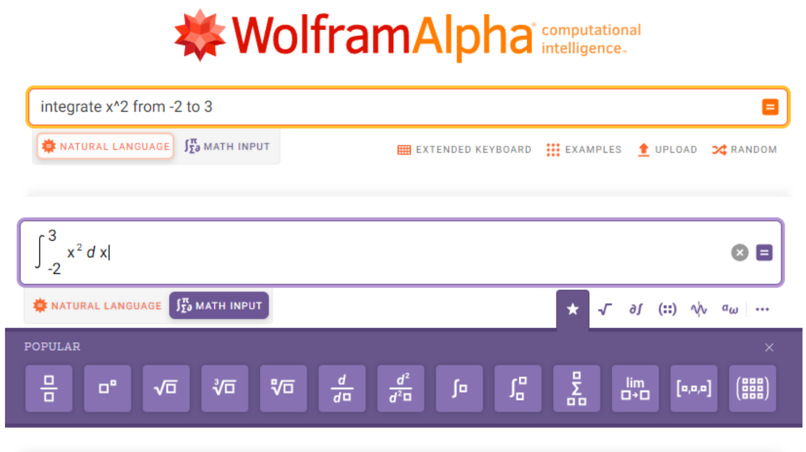

Figure 1: Wolfram|Alpha will accept inputs in either words or symbols.

- GeoGebra. Though Desmos has many of the same functions, an argument can be made that especially for applications involving revolution solids and surfaces more work has been done in GeoGebra. A quick web search will lead to a number of interactive tools like this one. It will be mostly useful to visualize things, but if you can do the calculations without then you can skip it.

3 The Rocket Equation

When building heavy machines that have to work against gravity, every choice has consequences. Aircraft and spacecraft are heavy, and one of the problems that arises from this is that it takes a lot of energy (in the form of fuel) to move up. However, adding more fuel is not a straightforward solution, as fuel adds mass and this in turn increases the amount of energy needed to move up. The velocity and mass both change over time and their rates of change are related. Konstantine Tsiolkovsky (Константин Циолковский), one of the first people to work on aeronautics, is credited with posing the problem in what we now know as the rocket equation. Since the relationship between $v(t)$ and $m(t)$ is already a complicated one, we assume that in space there are no external forces affecting our calculations.

Newton's formula $F = ma$ has a more general formulation. He described the force acting on an object as the rate of change of momentum ($p = mv$).

1. Write an equation that states Newton's second law of motion in terms of momentum p, mass m and velocity v. *Hint: "rate of change" in calculus means...*

   $F =$

2. Rewrite this equation allowing for m and v to be functions of time (i.e. $m(t)$ and $v(t)$. *Hint: F in general*

should be equal to the sum of two terms. Show that if the mass is constant, we get $F = ma$.

3. The law of conservation of momentum states that in a closed system, the total momentum remains constant. Interpret this result in terms of the derivatives above. If momentum is constant, then $F =$ ______.

4. In our reference scenario, $v'(t)$______0 and $m'(t)$______0, since the rocket starts at rest and the total mass of the rocket (with fuel) ________________ over time. This means that the two terms in the expression for F act in opposite directions.

5. We can interpret the two terms as corresponding to the two main contributors: the force of the rocket moving up and the force of the fuel burned being expelled down. Which one is which? Explain.

6. We will further simplify assuming the velocity of the burned fuel is constant: we will denote it with v_e, which stands for *exhaust velocity*. Since the mass is never zero, we may divide all terms by $m(t)$ to write

$$0 = \frac{v_e m'(t)}{m(t)} + v'(t)$$

   Integrate with respect to time between times t_i (initial time) and t_f (final time) to find an expression for $v(t)$ to show that

$$v(t_f) = v(t_i) - v_e \ln\left(\frac{m(t_f)}{m(t_i)}\right) = v(t_i) + v_e \ln\left(\frac{m(t_i)}{m(t_f)}\right)$$

7. We used a variation of this equation to model the velocity of a rocket launched from earth and using fuel at a constant rate of r mass units per time unit

$$v(t) = -gt - v_e \ln\left(\frac{m - rt}{m}\right)$$

How does this relate to the equation above? *Hint: Now that we're not in space, there may be external forces to consider.*

8. There is one assumption made about $v(t)$ above that means this setup only gives us partial information about where the rocket is at any given point in time. What oversimplification are we making? How can we make our results more realistic?

4 Design

The center of mass (also called *centroid*) of an object is what we usually draw when we depict forces acting on an object as if it were reduced to a single point. For two dimensional objects of uniform density, the centroid is a point where you can balance the object on a pin without it falling. You may remember computing these geometrically for triangles. Suppose a flat region R is bounded by $x = a$, $x = b$, $f(x)$ and the x-axis. In calculus, we can find the coordinates of the centroid $(\bar{x}, \bar{y})$ of R as

$$\bar{x} = \frac{1}{A} \int_a^b x f(x) dx$$

$$\bar{y} = \frac{1}{A} \int_a^b \frac{1}{2} [f(x)]^2 dx$$

where

$$A = \int_a^b f(x)dx$$

10. Show that for a circle with center $(0, 0)$, $\bar{x} = 0$ and $\bar{y} = 0$.

11. Let R be the triangle with vertices $(0, 0)$, $(5, 0)$ and $(2, 3)$. Find the centroid of R.

12. Let R be a semicircle with radius r. Find the centroid of R.

13. Let R be the region bounded by $4 - x^2$ and the x-axis. Find its center of mass.

14. Let R be the region bounded by $\sin(x)$ and the x-axis for $0 \leq x \leq \pi$. Find its center of mass.

15. Pappus of Alexandria developed a couple of theorems we'll find useful. One theorem states that if R is a plane region entirely on one side of a line l and R is rotated about l, then the volume of the resulting solid is the product of the area of R and the distance traveled by the centroid of R.

    (a) Use Pappus' theorem to show that the volume of a sphere with radius r is $\frac{4}{3}\pi r^3$.

(b) Use Pappus' theorem to find the volume of a torus with major radius R and minor radius r.

16. There is another theorem with an equivalent result for surface areas of revolution solids, but we'll need to determine the centroid of a curve first. Let C be the curve defined by $f(x)$ between $(a, f(a))$ and $(b, f(b))$ and let S be its length (using the arc length formula $S = \int_a^b ds$). Then the centroid of C is $(\bar{x}, \bar{y})$ where

$$\bar{x} = \frac{1}{S}\int_a^b x ds$$

$$\bar{y} = \frac{1}{S}\int_a^b f(x) ds$$

The theorem states that the surface area of the surface obtained by rotating C about the x-axis is equal to the product of the length of C and the distance traveled by the centroid of C.

(a) Use Pappus' theorem to show that the surface area of a sphere with radius r is $4\pi r^2$.

(b) Use Pappus' theorem to find the surface area of a torus with major radius R and minor radius r.

The center of mass in aeronautics is usually advised to be within certain ranges: when aircraft or spacecraft are loaded incorrectly, it can cause a lot of problems (instability, crashes, load shifting). We will examine the center of mass of aircraft with different approximate aerial views. Since the equations above only work in two dimensions, we will use an approximation of the aerial view of a design for our calculations.

16. Find an aerial view of a plane. Paste the image below (making sure to credit it!).

17. Planes usually have lateral symmetry, so to approximate the centroid we can assume one coordinate is directly on the axis of symmetry and focus on finding the other. Find a function (either $y = f(x)$ or $x = g(y)$, depending on how you oriented your picture) to fit the contour of one side of the plane. You may use online graphing tools, interpolations, or trial and error to find a combination of functions that more or less fits. Find a formula and graph your function using any available software/website and paste an image/screenshot here where both the function's graph and formula are visible.

18. Use this function to set up an expression to find the centroid of the plane. Discuss whether you would be able to compute this by hand or not and why. Find a way to compute or estimate the centroid and plot it. Does this look reasonable?

19. Right now we only have an approximation because we used a function that only roughly fits. Would you trust the centroid's coordinates to be accurate if we could find a perfect contour function? Discuss what assumptions we are making that may not match the real life plane and how likely they are to affect our calculations.

20. Tsiolkovsky wrote about using centripetal acceleration to create artificial gravity in space, which resulted in designs where space stations are built as rotating wheels. You may have seen versions of this in movies (like *Interstellar* or *Space Odyssey*) but to date none exist in real life. What are some of the challenges involved in bringing a spacecraft like this to orbit? Venture a guess as to why none have been built (yet).

5 Orbits

Johannes Kepler predicted that orbits of planets and satellites follow conic sections (circles, parabolas, ellipses, hyperbolas).

21. For each conic section, write a formula of a function that represents the equation in the first quadrant. Make sure to point out its domain.

22. For each conic section, set up an expression to denote the length of the arc within the first quadrant. Evaluate the ones that you can by hand using methods learned in class or integration tables as needed. For the ones you can't solve by hand, use online tools like Wolfram Alpha and paste screenshots of the results. You may need to add an extra page here.

The arc length of an ellipse is notoriously difficult to compute. In the case where the ellipse is a circle, we can find its value exactly. The eccentricity of an ellipse measures how close it is to being a circle (an eccentricity value of 0 makes a circle, and values less than 1 make an ellipse).

$$e = \frac{\sqrt{a^2 - b^2}}{2}$$

As the eccentricity of the ellipse increases, the estimates get worse and worse. We will compare a few different estimates for the perimeter of ellipses with increasing eccentricity. To make the math simpler, we will set $b = 1$ and let a be a multiple of b (e.g. $a = 2b$, $a = 6b$, etc.), which in turn creates lower and lower eccentricities. Credit for ideas used in this portion of the exercises to Matt Parker, who made a video on this topic.

To compute the error of our approximations we need to know the exact value of the perimeter and that's unattainable. Instead, we will use a *very* good approximation that can be obtained from sums of the form

$$p = 2\pi a \left(1 - \sum_{i=1}^{\infty} \frac{(2i)!^2}{(2^i \cdot i)^4} \cdot \frac{e^{2i}}{2i-1}\right)$$

where e here stands for the eccentricity (not Euler's constant!) The first few terms are

$$2a\pi \left[1 - \left(\frac{1}{2}\right)^2 e^2 - \left(\frac{1 \cdot 3}{2 \cdot 4}\right)^2 \frac{e^4}{3} - \left(\frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6}\right)^2 \frac{e^6}{5}\right]$$

23. (a) Use any tool to compute the first 5 partial sums (the sums up to $n = 1$, $n = 2$, . . ., $n = 5$).

(b) Let $h = \frac{(a-b)^2}{(a+b)^2}$. Compute the first 5 partial sums when using the formula

$$p = \pi(a+b) \sum_{n=0}^{\infty} \binom{0.5}{n}^2 h^n$$

whose first few terms are

$$\pi(a+b)\left(1 = \frac{1}{4}h + \frac{1}{64}h^2 + \frac{1}{256}h^3 + \dots\right)$$

(c) Compare the differences between consecutive terms for the two different formulas. An approximation "approaches the right answer" when these differences are very small. Which formula approaches the right answer more quickly? How do you know?

(d) Create a gold standard value to compare against: it won't be the exact value, but it will be close enough. For the formula you selected in the previous step, approximate p with $n = 7$. Write down as many decimal places as the tool you are using can output.

24. Make a table with columns $a = b$, $a = 2b$, $a = 3b$, $a = 4b$, and $a = 5b$. Make rows corresponding to the approximate perimeter of an ellipse computed using

  (a) Riemann sums with $n = 10, 20$ (two rows).

  (b) Simpson's rule with $n = 10, 20$ (two rows).

  (c) The approximation $\pi(a + b)$.

  (d) The approximation $2\pi\sqrt{\frac{a^2+b^2}{2}}$.

  (e) The approximation $\pi\left[3(a + b) - \sqrt{(3a + b)(a + 3b)}\right]$ (this one is due to Srinivasa Ramanujan).

25. Make a plot comparing the errors of the approximations for different values of e (there should be one line for each approximation formula, and e values should go on the horizontal axis). Paste the plot on a new page.

26. Which approximation is best? Justify your decision by considering the error and difficulty in computing each value.

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Rubric

	1	2	3	4	5
Calculus concepts	Lots of work missing/incorrect.	Some work missing/incorrect.	About half of the work is complete and correct.	Most explanations are complete and correct, using only the tools learned so far.	All explanations are complete and correct, using only the tools learned so far.
Discussion (counts double)	Lots of answers missing/incorrect.	Some answers missing/incorrect or the answers provided are very shallow.	Shallow analysis or several incorrect answers.	A few answers incorrect but most questions are answered in depth.	All correct and in depth answers.
Math work (by hand)	Many solutions are incorrect or incomplete	A few solutions are incorrect or incomplete	Some solutions are missing steps or have small errors	A few solutions are missing steps or have small errors	All solutions are correct and complete
Math work (using tools to compute and sketch)	Many solutions are missing/incorrect	Some solutions are missing/incorrect/incomplete	A few solutions are incorrect/incomplete	Most answers are correct and complete	All solutions are correct and complete.
Clarity	It is hard to read/follow the work	Some of the work is hard to read/follow	The organization/tidiness leaves room for improvement but is readable	The work is generally easy to read/follow	It is very easy to read/follow the work done