Engineering Calculus III
Mini Project 2: Drawing Boards

Lina Fajardo Gómez

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This project will explore some of the topics covered in class through applications and generalizations in surface design. Though CAD software will be mentioned, you do not need to be familiar with them to work on this project. There are text fields and empty space provided for you to type in and write in your answers. You are welcome to add more pages or work in a separate document if you find it easier, but please make sure your final submission is a single pdf file. If that pdf file includes the rubric page at the end of this document, you will even see exactly how your score was computed.

1 Bézier Curves

Bézier curves are very often used in computer-aided design (CAD) software. They are named after Pierre Bézier, a French engineer led the fields of solid, geometric and physical modelling, curve representation and manufacturing systems. A key feature of Bézier curves is that they are built around "control points." These allow us to stretch and bend the curve to smoothly and continuously approximate virtually anything, regardless of whether an exact parametrization doesn't exist or is algebraically complicated. This is why they are used so commonly. This is a very nice video explanation of the subject, though we'll be checking many of the properties by hand.

As we have seen previously, the line connecting points P and Q can be described by

$$\vec{r}(t) = \vec{r_0} + t\vec{v}$$

if we set $\vec{r_0} = \overrightarrow{OP}$ and $\vec{v} = \overrightarrow{PQ}$.

1. As t varies through all real numbers, we get an infinitely long line. Show that if we restrict t to the interval $[0, 1]$ (i.e. $0 \leq t \leq 1$) we get exactly the line segment connecting P and Q.

2. If we write $\vec{r_1} = \overrightarrow{OQ}$, show that the line segment from P to Q can also be described as $\vec{r}(t) = (1-t)\vec{r_0} + t\vec{r_1}$.

3. Read a bit about linear interpolations. Explain in your own words why the equation above is a kind of linear interpolation.

4. A linear Bézier curve is a curve of the form

$$\vec{B}(t) = (1-t)\vec{P_0} + t\vec{P_1}$$

connecting the endpoints of $\vec{P_0}$ and $\vec{P_1}$. Why is this equivalent to a linear interpolation?

5. Let $\vec{r_0}(t)$ and $\vec{r_1}(t)$ be vector functions. Describe what $\vec{r}(t) = (1-t)\vec{r_0}(t) + t\vec{r_1}(t)$ represents in words and help your explanation with a diagram.[1] This GeoGebra activity may help.

6. A quadratic Bézier curve is a function

$$\vec{B}(t) = (1-t)[(1-t)\vec{P_0} + t\vec{P_1}] + t[(1-t)\vec{P_1} + t\vec{P_2}], \quad 0 \leq t \leq 1.$$

Explain what the curve does in words (i.e. what happens as t increases from 0 to 1?).

[1]In pure math circles, this is known as a linear homotopy.

7. Show that the lines tangent to the quadratic Bézier at $\overrightarrow{P_0}$ and $\overrightarrow{P_2}$ intersect at $\overrightarrow{P_1}$.

8. The points $\overrightarrow{P_0}, \ldots, \overrightarrow{P_2}$ are called the *control points* of the quadratic Bézier curve. We will use $\overrightarrow{B}_{\overrightarrow{P_i}\overrightarrow{P_j}\overrightarrow{P_j}}$ to represent a quadratic Bézier with control points $\overrightarrow{P_i}$, $\overrightarrow{P_j}$, and $\overrightarrow{P_j}$. For a cubic Bézier curve, we need four. The equation is

$$(1-t)\overrightarrow{B}_{\overrightarrow{P_0}\overrightarrow{P_1}\overrightarrow{P_2}} + t\overrightarrow{B}_{\overrightarrow{P_1}\overrightarrow{P_2}\overrightarrow{P_3}}, \quad 0 \le t \le 1.$$

Show that this is equivalent to

$$\overrightarrow{B}(t) = \sum_{i=0}^{3} \binom{3}{i} (1-t)^{3-i} t^i \overrightarrow{P_i}.$$

9. Use the binomial theorem to write an expression for the nth degree Bézier with $n+1$ control points.

10. Graph the cubic Bézier with control points $P_0(4,1)$, $P_1(28,48)$, $P_2(50,42)$, and $P_3(40,5)$. Graph the segments P_0P_1, P_1P_2 and P_2P_3. Describe how the control points determine the behavior of the curve.

11. For a cubic Bézier curve, prove that the tangent line at P_0 passes through P_1 and that the tangent at P_3 passes through P_2.

2 Squircles

Squircles are hybrids of squares and circles, sometimes also known as super ellipses. You can read a bit about them here. Notoriously, Apple products and designs use squircles, and not just rectangles and squares with rounded off corners, in their designs. Once you see it, it is hard to unsee. It may even be relevant to your future work projects if you ever find yourself designing for them, whether it happens at the level of UI design or the surface modeling for computers, phones, or tablets.

12. We can write squircles as equations of the form

$$x^{2n} + y^{2n} = 1$$

where n is a positive integer. Find a parametrization for the nth squircle using t as a parameter. You may find the pseudocode and examples in the Coding Curves blog helpful.

13. Set up an integral for the length L_n of the nth squircle. Use computer software to approximate the length for different values of n. Make a guess for $\lim_{n\to\infty} L_n$.

14. Sketch several squircles for different values of n and compare their shapes. Does the limiting behavior of the shapes agree with the limit you found above for L_n? Explain your reasoning.

15. Read through the discussion here about squircles and their curvature. Why are squircles desirable in design? What are some of their most important features?

16. Fill in the gaps between the equations in figure 8.4 and those in figure 8.5 and figure 8.6. Write out the missing steps.

3 Continuity

This video discusses the continuity of splines, while this instruction manual discusses the continuity of surfaces.

17. Summarize as best you can what you understood about geometric continuity from the video and manual. Why is it important? How does it relate to what we have learned about in the calculus course? If any concepts or topics are unfamiliar, mention that as well. What did you have to look up to understand better?

18. A common feature of CAD software is something called zebra stripe analysis. You can see a blurb about it in an Autodesk manual here. Explain in your own words how the zebra stripes can help you visualize geometric continuity.

19. Discuss why geometric continuity might be a desirable feature in class A surfaces and what engineering applications would benefit from it.

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Rubric

	1	2	3	4	5
Calculus concepts	Lots of work missing/incorrect/requires more advanced tools	Some work missing/incorrect/requires more advanced tools.	About half of the work is complete and correct.	Most explanations are complete and correct, using only the tools learned so far.	All explanations are complete and correct, using only the tools learned so far.
Math work (worth double)	Many solutions are incorrect or incomplete	A few solutions are incorrect or incomplete	Some solutions are missing steps or have small errors	A few solutions are missing steps or have small errors	All solutions are correct and complete
Clarity	It is hard to read/follow the work	Some of the work is hard to read/follow	The organization/tidiness leaves room for improvement but is readable	The work is generally easy to read/follow	It is very easy to read/follow the work done
Technology use	No evidence of technology being used	Technology used but not clear how	Some instructional value in the screenshots	A couple steps are left to the imagination but overall clear	The work can be reproduced exactly from the pictures
Analysis	The answers are very shallow or a lot of them are missing.	Some questions are shallow or missing.	A few questions have incorrect answers or suggest not enough was learned from the offered resources.	Most answers are thorough and use the provided resources.	All answers are thorough and use the provided resources.