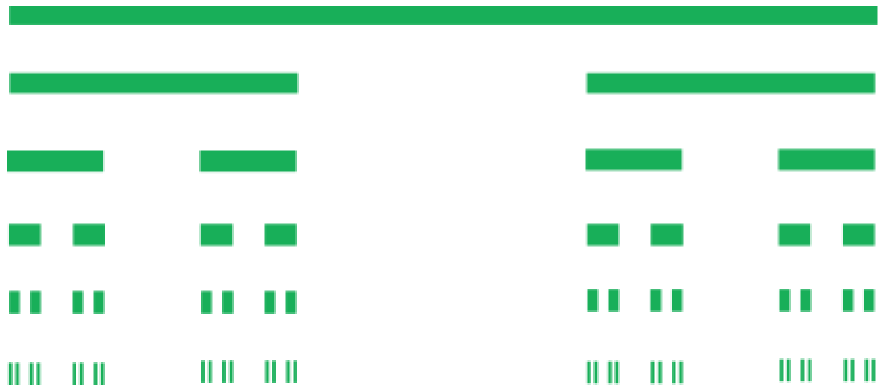

A BIG BOOK

WITH SO MANY PAGES

TO STUDY

Real Analysis

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credit where due (everywhere) to
Tom McKinley

[1]*(for like three or four proofs + all the help understanding and solving problems)*

Contents

Chapter 1

The Basics

Learning Objectives

The goal of this chapter is to familiarize the reader with:

1. Propositional logic
2. Proof writing
3. Elementary set theory
4. Functions
5. Cardinality of sets

Let's start at the very beginning[1] (a very good place to start). Much of the basics we present here are covered by courses designed to teach proof writing and mathematical thinking. Students who are already comfortable with the material may choose to skip it. The theory here is for those who could use the review without having to look elsewhere. The practice may provide a good opportunity to focus on good writing form.

1.1 Logic and Proof Writing

This section has no aspirations of being a complete introduction to logic. Rather, the goal is to ensure the language used throughout this document is clear, and that the reader may understand the structure of proofs. Logic

[1] To quote the famous approximation theorist E. A. Rakhmanov.

uses *statements*, which, unlike ordinary English sentences, we will informally define as sentences that must strictly be either true or false. If there's any question about why otherwise grammatically correct sentences may not be statements, consider: "This sentence is false." Really pause to think about it. Statements must be true or false so that we can either prove or disprove them. As a convention, we denote statements with P, Q, R, etc. and objects which may or may not satisfy the conditions in those statements by x, y, n, etc.

Statements and objects can be modified and combined through negations, connectors and quantifiers, which we translate to plain English in table 1.1. Rather than formally define these, we will rely on the reader's naive understanding of their meaning from experience and suggest that the notation is taken as shorthand.

Useful Logic Notation

Expression	Meaning
$\neg P$	the negation of P (sometimes $\sim P$)
$P \wedge Q$	P and Q
$P \vee Q$	P or Q
$P \implies Q$	if P then Q
	or P implies Q,
	or P is sufficient for Q,
	or Q is necessary for P
$P \iff Q$	P if and only if Q,
	or P and Q are equivalent,
	or P is necessary and sufficient for Q
$\exists x$	there exists an x
$\exists! x$	there exists a unique x
$\forall x$	for all x

Fact 1.1: Vacuously True Statements

Though counterintuitive, the reader would do well to remember that the statement

$$P \implies Q$$

is always true if P is false . We say in these cases that P is *vacuously true* (sometimes *trivially true*). For example, at the time of this writing, the statement "all Nobel prizes in mathematics have been awarded to me" is vacuously true.

Writing the negation of a statement will come in handy to construct counterexamples, prove results, and understand the nuances in definitions with slightly different wording.

Negations of Statements

Statement	Negation
$\neg(P \wedge Q)$	$\neg P \vee \neg Q$
$\neg(P \vee Q)$	$\neg Q \wedge \neg Q$
$\neg(P \implies Q)$	$P \wedge \neg Q$
$\neg\forall x$	$\exists x$
$\neg\exists x$	$\forall x$

Example 1.1.1: Negation

The negation of "$\forall x \exists y$ such that $P \implies Q$" is "$\exists x$ such that $\forall y$ $P \wedge \neg Q$." To prove that "for each prime number x there exists an even number y such that $x < 7$ implies $y = x$" is false, it suffices to show there exists a prime x such that for all even numbers y we have $x < 7$ but not $y = x$. We can take for example, $x = 5$.

It is worth pointing out that the order in which objects are presented and quantified matters (like the difference between uniform continuity and continuity), so we will review it here.

Example 1.1.2: Order of Quantifiers

The difference between $\forall x \exists y$ and $\exists y \forall x$ is demonstrated when analyzing the English sentences

"For all students there exists a teacher"

and

"There exists a teacher for all students"

In the former case, each student has a teacher (so there should be at least as many teachers as students). In the latter, a single teacher may teach all students.

Fact 1.2: Contrapositive Statements

The following two statements are logically equivalent:

$$P \implies Q$$

$$\neg Q \implies \neg P$$

We call the second statement the *contrapositive* of the first.

For easy reference, we list below some proof outlines inspired by the work in [Ham18], which can be a good reference for additional reading about proof writing. These can act as guides when trying to prove a particular result. Bear in mind that sometimes, in an exam setting, just being able to show that you knew *what* you have to prove, though perhaps not *how*, can demonstrate knowledge of the content being evaluated.

Outline for a Direct Proof

Proposition: $P \implies Q$

Proof.
Suppose P.
⋮
Then Q. □

In the simplest type of direct proof, all that is needed are the main definition referred to in the statement we are trying to prove and some elementary computation. In the example below, we use the outline and fill in the gap to prove a result.

Example 1.1.3:
Proposition: If n is even then n^2 is even.

Proof.
Suppose n is even.

Technique
Direct proof.

By definition, n is even $\iff \exists m$ such that $n = 2m$. Then

$$\begin{aligned} n^2 &= (2m)^2 \\ &= 4m^2 \\ &= 2(2m^2) \end{aligned}$$

Then n^2 is even. □

When a direct proof is not immediately apparent, it is worth trying to prove the contrapositive.

Outline for a Contrapositive Proof

Proposition: $P \implies Q$

Proof.
Suppose $\neg Q$.
⋮
Then $\neg P$. □

To illustrate how the contrapositive statement may be easier to prove than the original, consider the statement "If $n^2 - 4n + 5$ is even, then n is odd." To prove it directly we would have to write $n^2 - 4n + 5 = 2m$ which cannot readily be solved for n in order to draw a conclusion like $n = 2k + 1$. However, the contrapositive statement is straightforward with some computation.

Example 1.1.4:
Proposition: If $n^2 - 4n + 5$ is even, then n is odd.

Proof.
Suppose n is even.
By definition, n is even $\iff \exists m$ such that $n = 2m$. Then

Technique
Contrapositive proof.

$$\begin{aligned} n^2 - 4n + 5 &= (2m)^2 - 4(2m) + 5 \\ &= 4m^2 - 8m + 5 \\ &= 2(2m^2 - 4m + 2) + 1 \end{aligned}$$

Then $n^2 - 4n + 5$ is odd. □

A related technique is that of proofs by contradiction. We assume that the statement we are trying to prove is false, and follow that line of reasoning to a paradoxical result to conclude that our assumption must have been incorrect. "If I had won a Nobel prize I would have at least a million dollars, but my bank account says otherwise, so it must have been a dream."

Outline for a Proof by Contradiction

Proposition: R	**Proposition:** $P \implies Q$
Proof. For the sake of contradiction, suppose $\neg R$. ⋮ Then $C \wedge \neg C$, therefore R. □	*Proof.* For the sake of contradiction, suppose $P \wedge \neg Q$. ⋮ Then $C \wedge \neg C$, so $P \implies Q$. □

Advice
Choose the simplest proof.

Note that it is possible for the contradiction statement to be $P \wedge \neg P$, in which case it would have been better to proceed with a proof by contrapositive. In this case the proof by contradiction, while correct, is not the simplest approach and can be more difficult to follow. We present here a classic example of a proof by contradiction that motivates algebraic extensions of the set of rational numbers.

There is no single correct proof of a given proposition or theorem, and it is largely up to the reader to decide which method of proof is the best. A good proof should be easy to read and follow. It should offer enough but not excessive details and have a structure that makes the reasoning clear. It's a good idea to keep in mind that someone is eventually going to have to read your proof and decipher what you mean.

Proposition 1.3: $\sqrt{2}$ is Irrational

$\sqrt{2}$ is an irrational number.

Technique
Proof by contradiction.

Proof.
For the sake of contradiction suppose $\sqrt{2}$ is rational. By definition, a number is rational if and only if it can be written as $\frac{p}{q}$, for a pair of relatively prime integers p and q.

Then

$$\begin{aligned}\sqrt{2} &= \frac{p}{q}\\ 2 &= \frac{p^2}{q^2}\\ 2q^2 &= p^2\end{aligned}$$

This implies p^2 is even. If p were odd p^2 would also be odd, so p must be even and p^2 must be divisible by 4. This means that $2q^2$ is also divisible by 4, which implies that q is also even. Then we have that p and q are even and that they are relatively prime, a contradiction, so $\sqrt{2}$ is an irrational number. □

The simplest claims to prove are those that only have one statement. However, it is often necessary to have proofs nested inside of proofs or to combine several proof methods for a more complex result. A careful reading of the last example will make evident that an intermediate claim was used before reaching a contradiction: "if p were odd, p^2 would also be odd." At higher levels this is the kind of statement that does not require an explicit proof as a claim, but its purpose here is to show that there may be several steps in between the start and end of the outlines provided here. A good example of this is the proof of if and only if statements, which requires two smaller proofs.

Outline for an If And Only If Proof

Proposition: $P \iff Q$

Proof.
$P \implies Q$.
$Q \implies P$
⋮
Therefore $P \iff Q$. □

Remember that $P \iff Q$ is used to say that the statements are equivalent, so every definition stated as "(object) x is a y if (statement) P (is true)" should be read as x is a y *if and only if* P. On a related note, theorems that say statements are equivalent (where one of those statements is the definition of an object) can be read as a chain of implications. When one of the statements is a definition, the conclusion is that all equivalent statements are defining the same object.

To make a statement about all objects x satisfying some property, we write $P(x)$. This is useful when we want to make a statement about multiple objects, as outlined in the example below.

Example 1.1.5:
Let P be the property of being a prime number. Then $P(3)$ is true while $P(4)$ is false, as the number 3 is prime and the number 4 is not. To write a true statement about the fact that 4 is a composite number we can write $\neg P(4)$. Writing $\exists x \; : \; \neg P(x)$ is to say that there exists a number that is not prime. We can say something about the uniqueness of 2 as an even prime number with $\exists ! x \; : \; x$ is even $\wedge P(x)$. As a consequence, we can write $\forall x \; P(x) \wedge (x > 2) \implies x$ is odd.

When we make statements about a countable number of objects, a useful technique is that of proof by induction. There are two versions: weak and strong.

Outline for a Proof by Weak Induction

Proposition: $P(n)$ is true for all natural numbers n.

Proof.
Base case: $P(0)$ (sometimes $P(1)$).
⋮
Inductive step: $P(n) \implies P(n+1)$
⋮
Therefore $P(n)$ for all natural numbers n. □

Induction is often used to prove results about sequences or series, to prove inequalities, or to extend theorem results to countably infinite cases. We demonstrate it here on results related to function growth.

Example 1.1.6:
For any natural number n, if $n \geq 5$ then $2^n > n^2$. That is, exponential functions grow faster than their polynomial counterparts.

Proof.
We can prove this claim by induction.

Technique
Weak induction.

Base case: Let $n = 5$. We can readily check that

$$2^5 = 32 > 5^2 = 25$$

so the base case is true.

Inductive step: Suppose the statement holds for n. Then

$$\begin{aligned} 2^{n+1} &= 2 \cdot 2^n \\ &> 2n^2 \\ &= n^2 + n^2 \\ &= n^2 + 2n + n(n-2) \\ &\geq n^2 + 2n + 1 \\ &= (n+1)^2 \end{aligned}$$

Using the fact that $n(n-2) > 1$ whenever $n \geq 3$. Thus the statement holds for $n+1$.

As a result, the statement holds for all $n \geq 5$. □

The name "weak" has to do with the strength of the hypothesis in the inductive step and not the kind of result that can be proved. In weak induction we only need $P(n)$ to be true in order to prove $P(n+1)$, as opposed to the "stronger" hypothesis that $P(0) \wedge \cdots \wedge P(n)$ be true in strong induction. It is worth noting that when working with natural numbers, both induction principles are equivalent (if you can prove a result with one, you could prove it with the other). However, in some situations one may be easier to apply than the other.

We present below some useful results that apply variations of proof by induction, including one where the statement to be proved is a composite statement (of the $P(n) \implies Q(n)$ kind) and one where strong induction is used. First, an outline for a proof by strong induction:

Outline for a Proof by Strong Induction

Proposition: $P(n)$ is true for all natural numbers n.

Proof.
Base case: $P(0)$ (sometimes $P(1)$).
$\vdots$
Inductive step: $P(1) \wedge P(1) \wedge \cdots \wedge P(n) \implies P(n+1)$
$\vdots$
Therefore $P(n)$ for all natural numbers n. □

Definition: Means

We define the *arithmetic mean* of numbers $x_1, \ldots, x_n$ as

$$A = \frac{x_1 + \cdots + x_n}{n}$$

For nonnegative $x_1, \ldots, x_n$ the *geometric mean* is defined as

$$G = \sqrt[n]{x_1 \cdots x_n}$$

For positive $x_1, \ldots, x_n$ the *harmonic mean* is defined as

$$H = n \cdot \left(\frac{1}{x_1} + \cdots + \frac{1}{x_n} \right)^{-1}$$

The term "mean" here is used to indicate that these numbers fall between the smallest and largest of the x_i. As a consequence, appending A to the list of numbers $x_1, \ldots, x_n$ and recomputing the arithmetic mean results in A again. Similar results can be proved for G and H. Below, we prove that if the x_i are between values a and b, the arithmetic mean is too.

Proposition 1.4

Let $x_1, \ldots, x_n$ be numbers between a and b (i.e. $a \leq x_i \leq b$ for all i). Then

$$a \leq \frac{x_1 + \cdots + x_n}{n} \leq b$$

Proof.

We prove the claim by induction on n. Let $P(n)$ be the property $a \leq x_i \leq b$ for $1 \leq i \leq n$ and $Q(n)$ be the property $a \leq \frac{x_1+\cdots+x_n}{n} \leq b$. Let $S(n) = (P(n) \implies Q(n))$. We need to show that $S(1)$ holds and that $S(n) \implies S(n+1)$.

Technique

Nested direct proof in a proof by induction.

Base case: To check $S(1)$, we assume $P(1)$ and must show $Q(1)$. By definition, $P(1)$ holds if $a \leq x_1 \leq b$ and the average of the terms, $x_1/1 = 1$, lies between a and b, so $Q(1)$ holds as well. Therefore $P(1) \implies Q(1)$ and $S(1)$ is satisfied.

Inductive step: We assume $S(n)$ and want to show $S(n+1)$ is true, so we assume $P(n+1)$ (in addition to $S(n)$) and try to prove $Q(n+1)$ holds. Since $S(n)$ is true, in particular from $Q(n)$ we have

$$\begin{aligned} a &\leq \frac{x_1 + \cdots + x_n}{n} \leq b \\ an &\leq x_1 + \cdots + x_n \leq bn \end{aligned}$$

$P(n+1)$ implies $a \leq x_{n+1} \leq b$, so we can combine the inequalities to obtain

$$\begin{aligned} a + an &\leq x_1 + \cdots + x_n + x_{n+1} \leq b + bn \\ a(n+1) &\leq x_1 + \cdots + x_n + x_{n+1} \leq b(n+1) \\ a &\leq \frac{x_1 + \cdots + x_{n+1}}{n+1} \leq b \end{aligned}$$

which is $Q(n+1)$, as desired. Thus $S(n+1)$ holds and the proof is complete.

□

Strong induction can be used to prove what is known as the "AM-GM" inequality, which compares the arithmetic mean and the geometric mean. Conveniently, it also serves as an example for the proof of an if and only if statement. The inequality can be extended to include the harmonic mean, which we leave as an exercise for the reader using the same technique presented here.

Theorem 1.5: AM-GM Inequality

If $x_1, \ldots, x_n$ are nonnegative numbers, then

$$\sqrt[n]{x_1 \cdots x_n} \leq \frac{x_1 + \cdots + x_n}{n}$$

Equality holds if and only if $x_1 = \cdots = x_n$.

Technique

Proof by strong induction.

Proof.
Let $x_1, \ldots, x_n$ be nonnegative numbers, and let A and G denote their arithmetic and geometric means, respectively. Since the number n of terms is arbitrary, we use induction on k, where k is the number of terms different from A.

Base case: If $k = 0$, then all terms are equal to A and

$$\begin{aligned} \frac{x_1 + \cdots + x_n}{n} &= \frac{nA}{n} \\ &= A \\ &= \sqrt[n]{A^n} \\ &= \sqrt[n]{x_1 \cdots x_n} \\ &= G \end{aligned}$$

so the base case is true.

Inductive step: Suppose the statement holds for $1, 2, \ldots, k-1, k$ of the terms being different from A and consider the case where $k+1$ terms are different from A. Without loss of generality, we may assume the numbers are ordered so that $x_1 \leq \cdots \leq x_n$. Since at least two of these are different from A, using the averaging principle[2] yields $x_1 < A < x_n$.

[2] "Every set of numbers must contain a number at least as large and at least as small as the arithmetic average."

If $x_i = 0$ for any $i \in \{1, \dots, n\}$ then $G = 0$, so $G < A$ holds. We can therefore assume that all terms are positive. Let $y_1, \dots, y_n$ be defined by:

Trick

A stays the same but k is reduced to apply the induction hypothesis.

$$y_i = \begin{cases} A & i = 1 \\ x_1 + x_n - A & i = n \\ x_i & \text{otherwise} \end{cases}$$

Let A', G' denote the arithmetic and geometric means, respectively, of $y_1, \dots, y_n$. Note that $y_1 + \dots + y_n = x_1 + \dots + x_n$, so $A' = A$. Using the fact that $x_1 < A < x_n$ we have

$$\begin{aligned} (x_n - A)(A - x_1) &> 0 \\ Ax_n - A^2 + Ax_1 &> x_1x_n \\ A(x_1 + x_n - A) &> x_1x_n \end{aligned}$$

which yields

$$G' = \sqrt[n]{A \cdot x_2 \cdots (x_1 + x_n - A)} > G$$

In $y_1, \dots, y_n$ there are either k or $k - 1$ terms different from A, so we can apply the induction hypothesis to conclude that $G' < A$ and therefore $G < G' < A$.

□

1.2 Sets

The philosophical debate over what exactly is a set is beyond the scope of this document. A set is a concept that is impractical to formally define, so the intuition built from experience of what a set is will work fine for our intents and purposes. Without too much scrutiny of what each of the words may mean, we take a *set* A to be a collection such that an element x in "the universe" of whatever context we are working in can either belong to the collection (written $x \in A$) or not ($x \notin A$). We may also write $x \ni A$ (respectively, $x \not\ni A$, where appropriate. By convention, we use upper case letters like A, B, C for sets and lower case x, y, z, etc. for their elements.

The next few paragraphs are intended to provide a brief overview of set theoretical vocabulary and concepts, providing connections to the topics from the logic section where they might be useful. For additional reading on more formal set theory, see [Roi90]. To skip to a summary of the notation, see table 1.3.

It is possible to define a set by listing all of its elements: $A = \{0, 2, 4\}$. However, for arbitrary sets, we will most often default to *set builder notation*, where a set is described by the property shared by its elements. For example, the set of even numbers, rather than being listed as $\{0, 2, 4, \dots\}$ can be written as $\{m \mid m \text{ is a multiple of } 2\}$[3]. Relationships between sets should be understood as implications between the statements defining their elements.

The statement "every even number is an integer" is represented in the set containment

$$\{m \mid m \text{ is an even number}\} \subseteq \{n \mid n \text{ is an integer}\}$$

Since the sets are not equal, we say that the containment in this case is *proper* ($\subset$, or $\subsetneq$). In math slang, being an even number has "stronger" requirements than being an integer. If we describe A in terms of a property P satisfied by its elements, $A = \{x \mid P(x)\}$, and do the same for B with property Q, $B = \{x \mid Q(x)\}$, we can say $A = \{x \in B \mid P(x)\}$. In general, we say A is contained in B ($A \subseteq B$) if $x \in A \implies x \in B$ (see section 1.2).

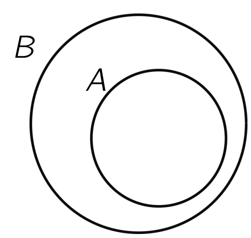

Figure 1.1: A is a subset of B.

We can easily make subsets this way by simply adding more requirements. For example with a different new restriction on the elements of B, we can write $\{x \in B \mid x \notin A\}$. This is called the *set difference* of B and A, denoted by $B \setminus A$ (see fig. 1.2). Using integers and even numbers for B and A, the set $B \setminus A$ is exactly the set of odd integers.

[3]Read as "all the elements m such that m is a multiple of 2."

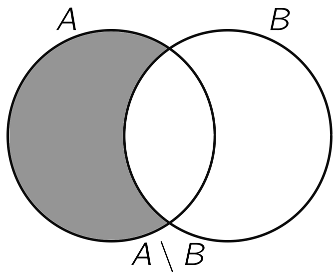

Figure 1.2: The set difference of A and B.

Technique
Proof of equivalent statements

If the sets were equal, then, we take that to mean that $x \in A \iff x \in B$, meaning that the property shared by elements of A is equivalent to that shared by elements of B. We could prove that $A = B$ using the outline for an if and only if proof by proving, separately, that $A \subseteq B$ and $B \subseteq A$.

Starting with sets A and B we can define new sets by combining or modifying the statements about their defining properties. If we want to consider elements that are either in A or B, $(x \in A) \vee (x \in B)$, we are describing the *union* of A and B: $A \cup B$ (see fig. 1.3). Saying that the set of integers is the union of even and odd numbers means each integer is either an even or an odd number.

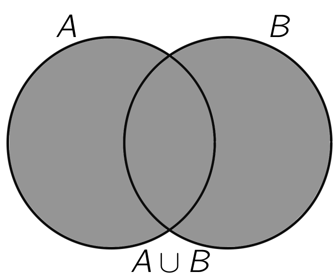

Figure 1.3: The union of A and B.

Changing the "or" in the description unions for an "and" yields the *intersection* of A and B, $A \cap B$ (see fig. 1.4). The set of numbers that are both even and odd is the *empty set* ($\varnothing$), as it has no elements. In this case (where the intersection is empty) we say the sets are *disjoint*. A straightforward way to obtain a set that is guaranteed to be disjoint from A is to consider the elements (in the "universe") that are *not* in A, i.e. $\neg(x \in A)$ define the *complement* of A, A^c.

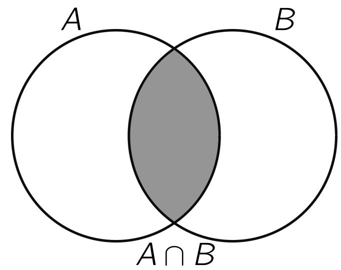

Figure 1.4: The intersection of A and B.

A way to combine elements of A and B is to list them as ordered pairs (a, b) where $a \in A$, $b \in B$. The set with all such pairs is the *Cartesian product* of A and B, denoted by $A \times B$. We may even use the product of multiple sets, or a set with itself multiple times, to define n-tuples instead of just ordered pairs.

The *power set* of A, written here as 2^A, is the set

$$2^A = \{B \mid B \subseteq A\}$$

For a finite set A, say, with n elements, we could explicitly list element down: $A = \{a_1, \ldots, a_n\}$. If we fix an arbitrary subset of A, B, each element of A may or may not be in B. For each a_i, assign either a 0 (if $a_i \notin B$) or a 1 (if $a_i \in B$). By counting all the possibilities, we obtain a total of 2^n possible subsets of A, including both A itself and $\varnothing$. Since $2^n > n$, we can see that the power set of A always has more elements than A. That is, the *cardinality* of 2^A ($|2^A|$) is greater than that of A ($|A|$). Though the proof is less straightforward for infinite sets the result holds true for all sets, so we list it here (and prove it later for countably infinite sets).

Advice
Use the techniques from this example to prove it.

Fact 1.6: Cardinality of the Power Set

For any set S, $|S| < |2^S|$.

To summarize the notation so far, with alternatives that may be found in other texts, we present the following table:

Useful Set Notation

Expression	Meaning
$x \in A$	x is an element of the set A
$x \notin A$	x is *not* an element of the set A

$\varnothing$	the empty set (sometimes $\{\}$)
$A \cup B$	the union of A and B
$A \cap B$	the intersection of A and B
A^c	the complement of A (sometimes A' or $\overline{A}$)
$A \setminus B$	the set difference of A and B (sometimes $A \cap B^c$)
$A \times B$	the Cartesian product of A and B
$A \subseteq B$	A is a subset of B (sometimes $A \subset B$)
2^A	the power set of A (sometimes $\mathcal{P}(A)$)
$\lvert A \rvert$	the cardinality of A (sometimes $\#A$)

Having defined some ways to make new sets from existing sets, let us prove a few results.

Proposition 1.7: DeMorgan's Laws

Let A, B, and C be arbitrary sets. Then

$$A \setminus (B \cup C) = (A \setminus B) \cap (A \setminus C)$$

and

$$A \setminus (B \cap C) = (A \setminus B) \cup (A \setminus C)$$

Proof.

For the first claim, we will use the definitions in terms of logical operators:

Technique

Equivalent statements used to prove equality of sets.

$$\begin{aligned}
x \in A \setminus (B \cup C) &\iff (x \in A) \wedge (x \notin (B \cup C)) \\
&\iff (x \in A) \wedge \neg((x \in B) \vee (x \in C)) \\
&\iff (x \in A) \wedge (\neg(x \in B) \wedge \neg(x \in C)) \\
&\iff ((x \in A) \wedge \neg(x \in B)) \wedge ((x \in A) \wedge \neg(x \in C)) \\
&\iff ((x \in A) \wedge (x \in B^c)) \wedge ((x \in A) \wedge (x \in C^c)) \\
&\iff (x \in A \cap B^c) \wedge (x \in A \cap C^c) \\
&\iff (x \in A \setminus B) \wedge (x \in A \setminus C) \\
&\iff x \in (A \setminus B) \cap (A \setminus C)
\end{aligned}$$

For the second claim, we will use double containment:

Technique

Double containment.

"$\subseteq$:"[4] Let $x \in A \setminus (B \cap C)$. Then by the definition of complement, $x \notin (B \cap C)$, which by the definition of intersection means $x \notin B$ or $x \notin C$. If $x \in A$ but $x \notin B$, then $x \in A \setminus B$. If $x \in A$ but $x \notin C$, then $x \in A \setminus C$. In any case, we have $x \in (A \setminus B) \cup (A \setminus C)$.

"$\supseteq$:" Let $x \in (A \setminus B) \cup (A \setminus C)$. Then by the definition of union, either $x \in A \setminus B$ or $x \in A \setminus C$). If $x \in A \setminus B$, then $x \in A$ and $x \notin B$. If $x \in A \setminus C$, then $x \in A$ and $x \notin C$. In both cases we have $x \in A$, and then either $x \notin B$ or $x \notin C$. If x fails to be in one of either B or C, by definition of the intersection, $x \notin B \cap C$. Therefore $x \in A \setminus (B \cap C)$. □

When we have a collection $\mathcal{A} = \{A_i \mid i \in I\}$ of sets with indices in I, we can consider the arbitrary union and intersection of the sets. We use $\bigcup_{i \in I} A_i$ or $\bigcup \mathcal{A}$ to denote the union of all the A_i, and similarly $\bigcup_{i \in I} A_i$ or $\bigcap \mathcal{A}$ to denote their intersection.

Before we can look at concrete examples, we need to establish some notation for the sets we will be working with. We will be working primarily with number sets, all of which can be endowed with algebraic operations. While it may be a bit of an oversimplification, to say that a ring is "a set that behaves like the integers with addition and multiplication" is a good way to get a feel for the most important properties from a concrete example.

Historically the sets were defined as extensions that allowed solutions to increasingly sophisticated polynomial equations (with natural numbers subtraction doesn't always make sense, in the set of integers division may not work, etc.) Their algebraic properties and axiomatic constructions will be alluded to but not expanded upon. The most important of these is the axiomatic description of the natural numbers (Peano's axioms) which are the foundation for proofs by induction. We leave these as optional reading (see [Lan66]).

Sets a real analysis student should be familiar with are:

1 $\mathbb{N}$: The set of natural numbers. We assume here that $0 \in \mathbb{N}$ and to denote the set of strictly positive natural numbers use $\mathbb{N}^*$. This is a countable[5] ordered set closed under addition. Great for proofs by induction, counting, and indexing.

[4]This is to say we are going to prove $A \setminus (B \cap C) \subseteq (A \setminus B) \cup (A \setminus C)$.

[5]A set whose elements can be listed, without leaving anything out, though the list may go on forever. i.e. A set with cardinality equal to $|\mathbb{N}|$.

2 $\mathbb{Z}$: The set of integers. This is the smallest extension of $\mathbb{N}$ that allows solutions to equations like $x+n=0$ where $n \in \mathbb{N}$. When paired with operators $+$ and $\cdot$ it forms a commutative ring with identity. Also countably infinite, it tends to serve the same purposes as $\mathbb{N}$.

3 $\mathbb{Q}$: The set of rational numbers. This is the smallest field containing $\mathbb{Z}$ (i.e. the smallest extension where $x \cdot n = 1$ has a solution for $n \in \mathbb{Z}$). Countable but dense[6], this set has many advantages over $\mathbb{N}$ and $\mathbb{Z}$ but falls short in two main aspects: polynomial equations (like $x^2 = 2$) may have no solutions in $\mathbb{Q}$ (so it is not algebraically closed) and not every bounded subset of $\mathbb{Q}$ has a supremum (or least upper bound) in $\mathbb{Q}$ (so it is not complete).

4 $\mathbb{R}$: The set of real numbers. This is the smallest extension of $\mathbb{Q}$ that has the leaves no "holes."[7] If $\mathbb{Z}$ can be visualized as evenly spaced dots on a line, and $\mathbb{Q}$ as tightly packed dots, so that there are dots between any two dots, $\mathbb{R}$ can be thought of as a solid line. Unlike its predecessors in the list this set is uncountably infinite[8] (a useful fact in cardinality-based arguments).

5 $\mathbb{C}$: The set of complex numbers. This is an algebraic extension of $\mathbb{R}$ that allows solutions to $x^2 + 1 = 0$. In many (but not all) ways, it is good to think of it as $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$.

6 $\mathbb{R}^n$: Euclidean n-dimensional space. Usually considered as a vector space over $\mathbb{R}$. This will be a good mental model to start understanding higher dimensional spaces.

Remark Interval Notation. In an ordered set X, we can write

$$\begin{aligned}
(a,b) &= \{x \in X \mid a < x < b\} \\
(a,b] &= \{x \in X \mid a < x \leq b\} \\
[a,b) &= \{x \in X \mid a \leq x < b\} \\
[a,b] &= \{x \in X \mid a \leq x \leq b\}
\end{aligned}$$

[6] Used here to mean that there are rational numbers arbitrarily close to any other number.

[7] This will later be called the least upper bound property

[8] Any way you try to list it, you will always fail to include all elements, even with unlimited time to go on listing.

The sets $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, and $\mathbb{R}$ are ordered, so we can unambiguously use interval notation in them. Note that in other resources (notably [Die60]) the interval (a, b) is denoted by $]a, b[$.

Example 1.2.1: Arbitrary Intersections and Unions

Consider the following sets as subsets of $\mathbb{R}$.

- Let $A_n = \{n, -n\}$, where $n \in \mathbb{N}$. Then $\bigcup_{n\in\mathbb{N}} A_n = \mathbb{Z}$.
- Let $A_{n,m} = \left\{\frac{n}{m}\right\}$ where $n \in \mathbb{Z}$ and $m \in (\mathbb{Z} \setminus \{0\}) = \mathbb{Z}^*$. Then $\bigcup_{(n,m)\in\mathbb{Z}\times\mathbb{Z}^*} A_{m,n} = \mathbb{Q}$.
- Let $A_n = \left[0, \frac{1}{n}\right]$, where $n \in \mathbb{N}^*$. Then $\bigcap_{n\in\mathbb{N}^*} A_n = \{0\}$.
- Let $A_n = (-n, n)$, where $n \in \mathbb{N}$. Then $\bigcup_{n\in\mathbb{N}} A_n = \mathbb{R}$.

Several other results that can be proved through similar methods to the ones we have presented so far are listed below.

Fact 1.8: Set Operations

$$\begin{aligned}
A \cap (B \cup C) &= (A \cap B) \cup (A \cap C) \\
A \cup (B \cap C) &= (A \cup B) \cap (A \cup C) \\
(A \cup B)^c &= A^c \cap B^c \\
\left(\bigcup A_i\right)^c &= \bigcap A_i^c \\
(A \cap B)^c &= A^c \cup B^c \\
\left(\bigcap A_i\right)^c &= \bigcup A_i^c \\
A \times (B \cap C) &= (A \times B) \cap (A \times C)
\end{aligned}$$

1.3 Functions

Functions are a central part of analysis (if not all of mathematics). Most readers will be familiar with functions through their graphs, but we wish to define them more formally (*read: abstractly*),as subsets of the Cartesian product of two sets representing a set of rules to associate their elements. We use functions to compare and relate mathematical structures, gain information about them , and to establish connections between previously unrelated objects. In short, functions can help us understand new objects by building on the ones we were already comfortable with.

Definition: Function

A *function* (sometimes *map* or *mapping*) $f : X \longrightarrow Y$ is a subset $f \subseteq X \times Y$ such that for each $x \in X$ there exists a unique ordered pair $(x, y) \in f$ containing x. We abbreviate $(x, y) \in f$ as $f(x) = y$. In effect, the equation $f(x) = y$ describes all pairs in f, so this rule defines the function. We call X the *domain* of f, the elements of X the *inputs* of f, Y the *range* of f, and elements of Y the *outputs* of f.

The conventional notation is to use letters like f, g, h for functions and the usual set conventions for their domain and range. We visually represent functions between arbitrary sets (with no particular structure) as arrows connecting elements of the sets (see fig. 1.5).

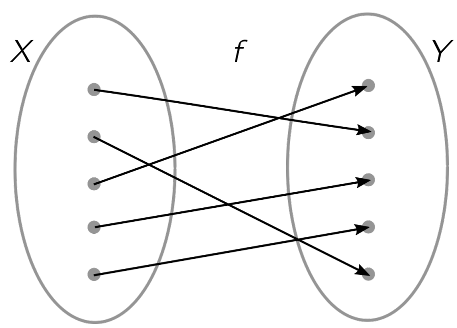

Figure 1.5: A function $f : X \longrightarrow Y$.

Example 1.3.1:

Another way to establish the uniqueness of a pair (x, y) for each $x \in X$ is to say that whenever $(x, y_1), (x, y_2) \in f$ then $y_1 = y_2$. Consider $f : \mathbb{N} \longrightarrow \mathbb{N}$ defined by the rule $f(x) = 2^x$. If $(3, 8)$ and $(3, m)$ are both in f, then we can deduce that $m = 8$, as only one value can be assigned to

e. In algebra, precalculus or calculus classes this property is known as the vertical line test (two values on the same vertical line would be two values assigned to the same input).

Definition: Image, Inverse Image

Let $f : X \longrightarrow Y$ be any function. The *inverse* of f is the set $f^{-1} = \{(f(x), x) \mid x \in X\}$. For $S \subseteq X$, the *direct image* of S is

$$f(S) = \{f(x) \mid x \in S\}$$

For $T \subseteq Y$, we define the *inverse image, or pre-image* of T as

$$f^{-1}(T) = \{x \in X \mid f(x) \in T\}$$

Example 1.3.2:
Let $f : \mathbb{Z} \longrightarrow \mathbb{N}$ be defined by $f(n) = n^2$, $A = \{-2, 3, 5\}$ and $B = \{4, 9, 16\}$. Then $f(A) = \{4, 9, 25\}$ and $f^{-1}(B) = \{2, -2, 3, -3, 4, -4\}$. The reader is encouraged to consider how sets such as $f(A \cup B)$, $f(A^c)$, or $f(f^{-1}(B))$ behave. How do do set operations behave when combined with the function or its inverse?

As we build a larger foundation of core concepts we will find more and more ways to relate them to one another. The image and inverse image can be combined with set operations to obtain a series of useful results.

Fact 1.9: Functions and Set Operations

Let $f : X \longrightarrow Y$ be any function, let $S, S_1, S_2, S_i \subseteq X$ for all $i \in I$, and $T, T_1, T_2, T_j \subseteq Y$ for all $j \in J$. Then

$$f\left(\bigcup_{i \in I} S_i\right) = \bigcup_{i \in I} f(S_i)$$

$$f\left(\bigcap_{i \in I} S_i\right) \subseteq \bigcap_{i \in I} f(S_i)$$

$$f(S_1 \setminus S_2) \subseteq f(S_1) \setminus f(S_2)$$

$$f^{-1}\left(\bigcup_{j\in J} T_j\right) = \bigcup_{j\in J} f^{-1}(T_j)$$
$$f^{-1}\left(\bigcap_{j\in J} T_j\right) = \bigcap_{j\in J} f^{-1}(T_j)$$
$$f^{-1}(T_1 \setminus T_2) = f^{-1}(T_1) \setminus f^{-1}(T_2)$$
$$S \subseteq f^{-1}(f(S))$$
$$T \supseteq f(f^{-1}(T))$$

Note that in many cases equality is not guaranteed without additional conditions on f, and that what may hold true for a finite number of sets may not hold for general unions or intersections. Many of these make great exercises, and can be proved with the techniques demonstrated so far.

To see that in general $f\left(\bigcap_{i\in I} S_i\right) \not\supseteq \bigcap_{i\in I} f(S_i)$ we need to find a function f and sets S_i such that for some $x \in \bigcap_{i\in I} f(S_i)$, $x \notin f\left(\bigcap_{i\in I} S_i\right)$. It is not necessary to go directly to a generalized union, and we can find counterexamples with just a couple of small sets. One way to ensure an element does not belong to $f(S_1 \cap S_2)$ is to make the set empty. It is possible, for example, for disjoint sets to have the same image. Consider $S_1 = \{1, 2, 3\}$, $S_2 = \{-1, -2, -3\}$ and $f(x) = |x|$. Then $f(S_1) = f(S_2)$, so $f(S_1) \cap f(S_2) = \{1, 2, 3\}$ while $S_1 \cap S_2 = \varnothing$ and therefore $f(S_1 \cap S_2) = \varnothing$.

Technique
Construct counterexamples

However, it was not necessary for the sets to be disjoint. Take for example $S_3 = \{-1, 0\}$ and $S_4 = \{0, 1\}$. Then $f(S_3) = f(S_4) = \{0, 1\}$ so $f(S_3) \cap f(S_4) = \{0, 1\}$ while $f(S_3 \cap S_4) = f(0) = 0$. What both constructions have in common and makes equality fail is the fact that the images of the sets may overlap more than the sets do. That is, points that the sets don't share in common can map to the same values. This makes the intersection of the images comparatively "big," while the the intersection of the sets (and therefore its image) may be "small."

When working through exercises it may be easy to default to functions that are familiar. The flaw with this method, which is good for scratch work and to get a rough idea of what is going on at first, is that often these fall into categories of functions that are all "the same." Likely the function types that immediately come to mind include polynomial, absolute value, radical,

Advice
Challenge the first examples that come to mind: are they truly general?

exponential, logarithmic, and trigonometric.[9] It is a good idea to get into the habit of thinking of the most general function possible: not just a polynomial, not just applied to a finite set. You should really challenge your notion of what a function is, as the construction of counterexamples is a great skill to have and will require a lot of "outside the box" thinking.

Definition: One-to-one, Onto

Let $f : X \longrightarrow Y$ be a function. We say that f is *one-to-one*[10], or *injective*, if for all $x_1, x_2 \in X$ $f(x_1) = f(x_2) \implies x_1 = x_2$. We call f *onto*, or *surjective*[11], if for all $y \in Y$ there exists an $x \in X$ such that $f(x) = y$.

A function fails to be one-to-one if it more than one point maps to the same output, and fails to be onto if any point in the domain fails to have an arrow pointing to it (see fig. 1.6).

Remark. Though the notation will not be used here, readers should be aware that other texts may use the shorthand $f : X \hookrightarrow Y$ to denote a one-to-one function, $f : X \twoheadrightarrow Y$ for an onto function.

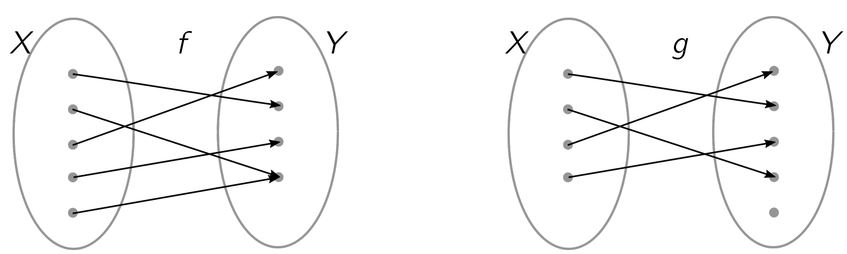

Figure 1.6: f is not one-to-one, and g is not onto.

Note that the definition for f being one-to-one is equivalent to requiring $f^{=1}$ to be a function. Seen this way, the uniqueness of outputs translates into a "horizontal line test," analogous to the vertical line test used to determine if an arbitrary set of pairs is a function. If an injection from X into Y exists, we can see X as being "a part of"[12] Y.

[9] Kudos to you if you have one in mind that hasn't been mentioned yet, like constants, piecewise functions, step functions, or pulse functions.

[10] This naming convention was standardized by Bourbaki. Read more here.

[12] A better word for this is *embedding*, but a proper embedding must also preserve the structure of X and Y, which we haven't defined the language for yet.

Example 1.3.3: One-to-one Functions

The function $f : \mathbb{R} \longrightarrow \mathbb{R}$, defined by $f(x) = x^3$ is one-to-one. Let $x_1, x_2 \in \mathbb{R}$. Then $f(x_1) = f(x_2)$ implies $x_1^3 = x_2^3$, so that $x_1^3 - x_2^3 = (x_1 - x_2)(x_1^2 + x_1x_2 + x_2^2) = 0$. This equation has a solution when either $x_1 = x_2$ or $x_1 = \frac{-x_2 \pm \sqrt{3}ix}{2}$ However, in the second case the solutions are not real, so it must be the case that $x_1 = x_2$. On the other hand, the function $g : \mathbb{R} \longrightarrow \mathbb{R}$, defined by $g(x) = x^2$ is not because, for example, $g(-1) = g(-1) = 1$.

While a function is expected to use every point in its domain in at least some pair, the same may not be true of every point in its range. We interpret as surjectivity that the entire range is "covered" by the function. often, if it is desired, it can be forced by restricting the range. In many notable and less obvious examples, though, whether or not this is possible is tied to a comparison of the cardinalities of the sets involved.

Example 1.3.4: Onto Functions

The function $f : \mathbb{R} \longrightarrow \mathbb{R}$ defined by $f(x) = e^x$ is not surjective. For example, there is no value of x that maps to $0 \in \mathbb{R}$. However, $g : \mathbb{R} \longrightarrow (0, \infty)$ defined by $g(x) = e^x$ is surjective, as for any $y \in (0, \infty)$ $\ln(y)$ is well defined and satisfies $g(\ln(y)) = y$. In general $f : X \longrightarrow f(X)$ is always onto, though not always interesting to consider. No function $h : \mathbb{N} \longrightarrow \mathbb{R}$ can be onto and the reason why (which boils down to the fact that $|\mathbb{R}| > |\mathbb{N}|$) says more about the differences between the sets than a function onto some countable subset of $\mathbb{R}$ would.

Mostly to establish some common notation, we introduce here ways of combining functions so that we may prove results with them later.

Definition: Restriction, Composition

Let $f : X \longrightarrow Y$, $g : Y \longrightarrow Z$ be arbitrary functions and let $A \subseteq X$. The *restriction* of f to A, denoted $f|_A$, is the map $f|_A : A \longrightarrow Y$ defined by $f|_A(x) = f(x)$ for each $x \in A$. The *composition* of g and f, denoted $g \circ f$[13], is the function $g \circ f : X \longrightarrow Z$ defined by $(g \circ f)(x) = g(f(x))$.

[-1] Read as "g following f" or "g of f."

Beyond restricting the range, we can restrict the domain of a function to make it a bijection. For example $f : \mathbb{R} \longrightarrow [0, \infty)$ defined by $f(x) = x^2$ is not a bijection but $f|_{[0,\infty)}$ is. Restrictions and compositions can inherit some properties of their parent functions.

Fact 1.10: Composition of Functions

Let $f : X \longrightarrow Y$, $g : Y \longrightarrow Z$. Then

1. If f and g are one-to-one, so is $g \circ f$.
2. If f and g are onto, so is $g \circ f$.
3. If f and g are bijections, so is $g \circ f$.

With this new vocabulary we can amend our previous list of useful facts to specify when equality holds.

Fact 1.11: Functions and Set Operations

Let $f : X \longrightarrow Y$ be any function, let $S, S_1, S_2, S_i \subseteq X$ for all $i \in I$, and $T \subseteq Y$. If f is one-to-one, the following results hold:

- $f\left(\bigcap_{i\in I} S_i\right) \subseteq \bigcap_{i\in I} f(S_i)$
- $f^{-1}(f(S)) = S$

If f is onto, the following result holds:

- $f(f^{-1}(T)) = T$

Though the inverse of a one-to-one function $f : X \longrightarrow Y$ is also a function, it may not be well-defined as not every point in Y may be in one of the pairs. We say that $g : Y \longrightarrow X$ is a *left inverse* of f if $g(f(x)) = x$ for all $x \in X$. If f is one-to-one, $g = f^{-1}$ is its left inverse. In other words: if only one x satisfies $f(x) = y$, we can recover x from y because no information is lost by f. Let us consider $f : \mathbb{R} \longrightarrow \mathbb{R}$ defined as $f(x) = x^2$. Since $(-2)^2 = 2^2$, it is impossible to tell whether we started at -2 or at 2 when we say $x^2 = 4$.

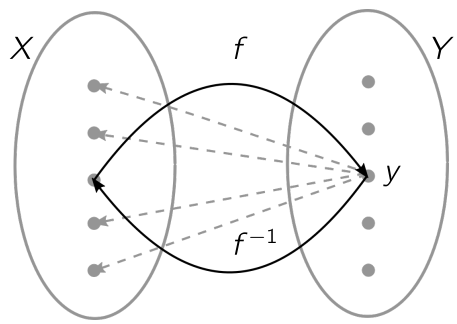

Figure 1.7: If there were multiple choices for x such that $f(x) = y$, f^{-1} may not recover the correct one when composed with f.

Similarly, we can define a $g : Y \longrightarrow X$ to be a *right inverse* of f if for each $y \in Y$ we have $f(g(y)) = y$. If f is onto, $g = f^{-1}$ is its right inverse. This works because when f is onto we can travel back from any y. The fact that applying f again yields y is guaranteed, even if there are multiple possible assignments for $g(y)$. To continue with the example of $f(x) = x^2$, whether we assign $g(4) = 2$ or $g(4) = -2$, we will have $f(g(4)) = 4$. Thus for a

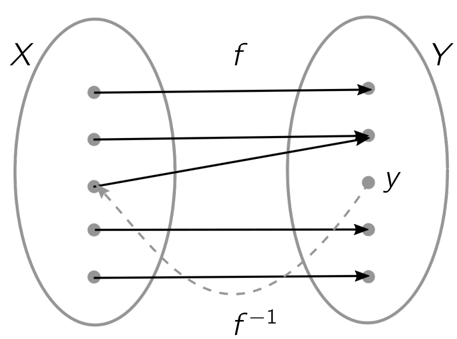

Figure 1.8: If a point $y \in Y$ is not mapped to, the inverse is not defined at every point in its domain.

function to be invertible, in the sense that it's possible to go back and forth between inputs and outputs without losing information, we require it to be both one-to-one and onto. We will in the future refer to invertible functions as bijections, which we define below.

Definition: Bijection, Invertible

A function that is both injective and surjective is said to be a *bijection*. A function $f : X \longrightarrow Y$ is said to be *invertible* if there exists a function $g : Y \longrightarrow X$ such that $(f \circ g)(y) = y$ for all $y \in Y$ and $(g \circ f)(x) = x$ for all $x \in X$.

A bijection can be read as a simple equivalence between sets (or at least between their sizes). With bijections defined, we can formalize the notion of countable sets.

Definition: Cardinality, Countable Sets

Two sets A, and B have the same *cardinality* if there exists a bijection between A and B. A is said to be *countable* if $|A| = |\mathbb{N}|$.

Example 1.3.5: Union of Countable Sets

Let A, B be countable sets. Since there exist bijections between A and $\mathbb{N}$, we can write $A = \{a_0, a_1, a_2, \ldots\}$. Similarly, $B = \{b_0, b_1, b_2, \ldots\}$ We can define a bijection between $\mathbb{N}$ and $A \cup B$ as follows:

$$f(n) = \begin{cases} a_i & n = 2i \\ b_i & n = 2i + 1 \end{cases}$$

The proof that this is indeed a bijection is left as an exercise to the reader.

This proves that a finite union of countable sets is again countable. As we impose more structure on sets and properties on functions to preserve that structure, the comparisons we are able to make with bijections will become more sophisticated.

Example 1.3.6: Bijections

The function $f : \mathbb{R} \longrightarrow \mathbb{R}$ defined by $f(x) = (x - 1)(x - 2)(x - 3)$ is onto but not one-to-one (this can be readily seen from its graph so we omit a proper explanation). The function $g : \mathbb{N} \longrightarrow \mathbb{R}$ defined by $g(x) = x$ is one-to-one but not onto. The function $h : \mathbb{R} \longrightarrow (0, 1)$ defined by

$$h(x) = \frac{e^x}{e^x + 1}$$

is a bijection. It is an one-to-one because $h(x_1) = h(x_2)$ yields

$$\begin{aligned} \frac{e^{x_1}}{e^{x_1}+1} &= \frac{e^{x_2}}{e^{x_2}+1} \\ e^{x_1}(e^{x_2}+1) &= e^{x_2}(e^{x_1}+1) \\ e^{x_1} &= e^{x_2} \\ x_1 &= x_2 \end{aligned}$$

It is surjective as for $y \in (0,1)$ we have

$$\begin{aligned} h\left(\ln\left(\frac{-y}{y-1}\right)\right) &= \frac{\frac{-y}{y-1}}{\frac{-y}{y-1}+1} \\ &= \frac{\frac{-y}{y-1}}{\frac{-y+y-1}{y-1}} \\ &= y \end{aligned}$$

This result highlights the fact that any open interval in $\mathbb{R}$ is "as big" as the entire real number line.

Though it is intuitively easier to imagine that a bijection exists between $(0,1)$ and $[0,1]$, it is not as easy to explicitly construct one. The result is less important than the techniques used in this construction, so we will add comments in the proof to explain why some simpler approaches would not work.

Proposition 1.12

There exists a bijection $f : (0,1) \longrightarrow [0,1]$.

Proof.
The two sets correspond almost exactly to each other and only two points (0 and 1 in $[0,1]$) would be unaccounted for if we mapped $x \mapsto x$. This could even be adapted to map $(0,1)$ to a more general interval like (a,b) by mapping $x \mapsto (b-a)x + a$. Fundamentally, the flaw in this approach is that it leaves no options in the domain to map to the endpoints of the closed interval. If every point of $(0,1)$ is accounted for and we haven't assigned any of them to map to 0 or 1 there's nothing left to choose.

Advice
Examine carefully what makes a proposed solution fail.

What about picking some points $x, y \in (0,1)$ to map to 0 and 1? If we did, we would have to make a correspondence between the remaining points mapping $(0,x) \cup (x,y) \cup (y,1)$ to $(0,1)$. However, an identity function also fails here as now there are two inner points of $(0,1)$ that will not be covered.

For any finite number of segments we try to divide the intervals into, there will always be some point in the range that is not covered. Key word "finite." With an infinite number of segments, finding one or two more points won't be an issue.

Trick
Map countable subset to countable subset and intervals to intervals.

Any countable subset of $(0,1)$ will work, but we will choose $A = \{\frac{1}{n+1} \mid n \in \mathbb{N}^*\}$. We can map A to an also countable subset $A' = \{0,1\} \cup A$ of $[0,1]$. Since they are both countable, it is sensible to expect a bijection there. As constructed, $(0,1) \setminus A = [0,1] \setminus A'$ so that we can use the identity function outside of A. We define a bijection $f : (0,1) \longrightarrow [0,1]$ as follows

$$f(x) = \begin{cases} 0 & x = \frac{1}{2} \\ 1 & x = \frac{1}{3} \\ \frac{1}{n+2} & = x = \frac{1}{n+1}, n > 2 \\ x & \text{otherwise} \end{cases}$$

The first three cases map A to A', while the remaining points are fixed. We will now carefully check that this is indeed a bijection.

To see that f is one-to-one, let $x_1, x_2 \in (0,1)$ and suppose $f(x_1) = f(x_2) = y$. There are two cases to consider: either $y \in A'$ or $y \notin A$. If $y \in A'$, then one of the following is true:

- $y = 0$, in which case $x_1 = x_2 = \frac{1}{2}$
- $y = 1$, in which case $x_1 = x_2 = \frac{1}{3}$
- $y = \frac{1}{n+2}$ for some $n > 2$, in which case $x_1 = x_2 = \frac{1}{n+1}$

If $y \notin A'$, then $x_1 = x_2$ directly follows from the construction of f. We can therefore deduce that f is one-to-one.

To see that f is onto, consider an arbitrary $y \in [0,1]$. The cases outlined above exactly describe the possibilities for the $x \in (0,1)$ that satisfy $f(x) = y$. Thus f is onto.

With these two requirements met, we can conclude that f is a bijection.

□

1.4 ⋆Factorials and Binomial Coefficients

While not strictly in line with the other learning objectives in this chapter, the topic of factorials and binomial coefficients provides excellent opportunities to see new uses of the techniques discussed. While they are primarily used for counting (i.e. in combinatorics) they will be featured, for example, in work with sequences and their limits. They also open the door to a discussion of function growth, and a study study of comparisons between different function types (polynomial, exponential, factorial, and others). While function growth is of interest in its own right, we introduce it here through examples that may be more familiar to the reader.

Definition: Factorial

For a natural number n we define $n!$ (read "n factorial") by

- $0! = 1$
- $(n+1)! = n! \cdot (n+1)$

It is worth pointing out that $n!$ grows very fast, even compared to exponential functions. We state a version of the result here comparing $n!$ to 2^n, though similar statements can be made for other exponential functions.

Fact 1.13

Let $n \geq 5$. Then $n! > 2^n$.

Factorial expressions can be generalized to non-discrete domains add reference to future chapter. For the time being, we can add another definition that will be useful: binomial coefficients.

Definition: Binomial Coefficients

For $n, k \in \mathbb{N}$, $k \leq n$, we define the *binomial coefficient*

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

The expression $\binom{n}{k}$ is read as "n choose k" and counts the number of ways of choosing k out of n elements.

We will use the following result about binomial coefficients to prove the binomial theorem.

Proposition 1.14

Let $n, k \in \mathbb{N}$ where $k \leq n-1$. Then

$$\binom{n-1}{k-1} + \binom{n-1}{k} = \binom{n}{k}$$

Proof.

$$\begin{aligned}
\binom{n-1}{k-1} + \binom{n-1}{k} &= \frac{(n-1)!}{(k-1)![(n-1)-(k-1)]!} + \frac{(n-1)!}{k![(n-1)-k]!} \\
&= \frac{(n-1)!k}{(k-1)!k(n-k)!} + \frac{(n-1)!(n-k)}{k!(n-k-1)!(n-k)} \\
&= \frac{(n-1)!}{k!(n-k)!}(k+n-k) \\
&= \frac{n!}{k!(n-k)!} \\
&= \binom{n}{k}
\end{aligned}$$

□

Theorem 1.15: Binomial Theorem

For numbers $x, y \in \mathbb{R}$ and $n \in \mathbb{N}$

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

Proof.

Technique
Weak induction.

We can prove this result by induction on n.

Base case: If $n = 0$, we have $(x+y)^0 = 1 = \binom{0}{0}$, so the statement is true.

Inductive step: Suppose the statement is true for n.

$$\begin{aligned}
(x+y)^{n+1} &= (x+y)(x+y)^n \\
&= (x+y)\left(\sum_{k=0}^{n}\binom{n}{k}x^k y^{n-k}\right) \\
&= \sum_{k=0}^{n}\binom{n}{k}x^{k+1}y^{n-k} + \sum_{k=0}^{n}\binom{n}{k}x^k y^{n-k+1} \\
&= \sum_{k=1}^{n}\binom{n}{k-1}x^k y^{n-(k-1)} + \sum_{k=0}^{n}\binom{n}{k}x^k y^{n-(k-1)} \\
&= \sum_{k=1}^{n}\underbrace{\left(\binom{n}{k-1}+\binom{n}{k}\right)}_{=\binom{n+1}{k}\text{ by proposition 1.14}} x^k y^{n-(k-1)} + y^{n+1} \\
&= \sum_{k=1}^{n}\binom{n+1}{k}x^{k+1}y^{(n+1)-k}
\end{aligned}$$

So the statement holds for $n+1$.

We therefore conclude that the equation is true for all $n \in \mathbb{N}$. □

Learning Checklist

Mastery of this chapter means being able to:

- ☐ Write statements using the correct notation (logic, sets, number sets)
- ☐ Identify equivalent statements (see, for example contrapositives)
- ☐ Understand and use proof outlines
  - ☐ direct proof
  - ☐ contrapositive proof
  - ☐ proof by contradiction
  - ☐ if and only if proof
  - ☐ proof by induction
- ☐ Identify vacuously true statements
- ☐ Decide whether one set is contained in another (and prove it)
- ☐ Know how function images and preimages behave when combined with set operations
- ☐ Determine whether a function is one-to-one, onto, or a bijection (and prove it)
- ☐ Compare the cardinalities of sets
- ☐ Construct examples and counterexamples

Practice

Memory

A. Recall definitions to complete the sentences.

1) The statement $P \implies Q$ is vacuously true if
2) $P \implies Q$ is equivalent to
3) $A \subseteq B$ if
4) $A = B$ if
5) $x \in A^c$ if
6) $x \in A \setminus B$ if
7) A function $f : X \longrightarrow Y$ is a subset $f \subseteq X \times Y$ such that
8) A function $f : X \longrightarrow Y$ is one-to-one if
9) A function $f : X \longrightarrow Y$ is onto if
10) A function $f : X \longrightarrow Y$ is a bijection if

Computation

B. Use set and logic notation to (re)write the statements.

1) Let P and Q be the statements defined by:

$$P: \text{ we wear pink}$$
$$Q: \text{ today it is Tuesday}$$

"On Tuesdays, we wear pink."

2) Let P, Q, and R be the statements defined by:

$$P: \text{ There exist real numbers that are not rational}$$
$$Q: \text{ Every integer s a rational number}$$
$$R: \text{ There exist real numbers that are not integers}$$

Write one true statement combining all three.

3) A number m is divisible by three if and only if there exists some integer k such that $m = 3k$.
4) All natural numbers are integers.
5) Every element of set A is positive.
6) There exists a subset A of X with 5 elements.
7) $f \subseteq X \times Y$ is a function.
8) $f : X \longrightarrow Y$ is one-to-one.
9) $f : X \longrightarrow Y$ is onto.

C. Describe the set that results from the given operations.

1) Let $A = \{0, 2, 4, 6\}$, $B = \{0, 3, 6\}$. Find $A \cap B$, $A \cup B$, $A \setminus B$.
2) Let $A = \{1, 2, 3, 4\}$. Explicitly list all the elements of 2^A. How many elements does 2^{2^A} have?
3) Let $2\mathbb{Z} = \{2n \mid n \in \mathbb{Z}\}$, $3\mathbb{Z} = \{3n \mid n \in \mathbb{Z}\}$. Find $2\mathbb{Z} \cap 3\mathbb{Z}$, $2\mathbb{Z} \cup 3\mathbb{Z}$, $2\mathbb{Z} \setminus 3\mathbb{Z}$.
4) Let $\mathbb{Q}(\sqrt{2})$ be the set $\{p+\sqrt{2}q \mid p, q \in \mathbb{Q}\}$. Find $\mathbb{Q}\cap\mathbb{Q}(\sqrt{2})$, $\mathbb{Q}\cup\mathbb{Q}(\sqrt{2})$ and $\mathbb{Q}(\sqrt{2}) \setminus \mathbb{Q}$.
5) Define the sets $A_n = \left(\frac{-1}{n}, \frac{1}{n}\right)$. Find $\bigcap_{n\in\mathbb{N}^*} A_n$ and $\bigcup_{n\in\mathbb{N}^*} A_n$.
6) Let $f : \mathbb{Z} \longrightarrow \mathbb{N}$ be defined by $f(n) = n^2$. Consider the sets $2\mathbb{Z} = \{2n \mid n \in \mathbb{Z}\}$, and $3\mathbb{Z} = \{3n \mid n \in \mathbb{Z}\}$. Find $f(2\mathbb{Z})$, $f(3\mathbb{Z})$, $f(2\mathbb{Z})\cap f(3\mathbb{Z})$ and $f(2\mathbb{Z} \cap 3\mathbb{Z})$.
7) Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be defined by $f(x) = 2^x$. Find $f^{-1}(\{4, 8, 16\})$, $f(\{0, -2, 4\})$.

Quick Applications

D. Use definitions to prove the desired result.

1) In the proof of theorem 1.5 we used $k = 0$ as the base case, but it was possible to start with $k = 1$. Prove that the inequality is vacuously true in this case. *Hint: Prove that it is not possible to have exactly one term different from A.*

2) Use the contrapositive of the definition of a one-to-one function to write an equivalent definition.

3) Write an outline for the proof of the fact that a function is invertible if and only if it is a bijection.

4) Prove that $A \times (B \cap C) = (A \times B) \cap (A \times C)$.

5) Let A, B be finite sets. Describe all possible functions from A to B. How many functions are there?

6) Let $f : X \longrightarrow Y$ be any function and let $S \subseteq X$ and $T \subseteq Y$. Prove that $S \subseteq f^{-1}(f(S))$ and $T \supseteq f(f^{-1}(T))$.

7) Let X and Y be finite sets. Prove that if $|X| > |Y|$, no function $f : X \longrightarrow Y$ can be one-to-one.

8) Let $f : X \longrightarrow Y$, $g : Y \longrightarrow Z$. Then

   i) If f and g are one-to-one, so is $g \circ f$.

   ii) If f and g are onto, so is $g \circ f$.

   iii) If f and g are bijections, so is $g \circ f$.

Focus on Technique

E. Use the techniques demonstrated throughout the chapter to prove the following results.

1) For arbitrary sets A and B, How would you prove that $A \not\subseteq B$?

2) Write the outline of a proof of the fact that a function $f : X \longrightarrow Y$ is onto.

3) Suppose $u_0, u_1, \ldots$ are numbers such that $u_{n+1} = u_n + u_{n-1}$, $u_0 = 0$ and $u_1 = 1$. Which type of induction would you use to prove that $u_n < 1.7^n$ for all n?

4) Let $x_1, \ldots x_n$ be non-negative numbers. Prove that the harmonic mean of the numbers is less than or equal to the geometric mean.

5) Use induction to prove that $1 + 2 + \cdots + n = \frac{n(n+1)}{2}$

6) Prove Bernoulli's inequality: if $a \geq -1$ then for every $n \in \mathbb{N}$ we have $(1 + a)^n \geq 1 + na$.

7) Construct an explicit bijection between $\mathbb{N}$ and $\mathbb{Z}$.

8) Construct an explicit bijection between $(-1, 1)$ and $\mathbb{R}$.

9) Let $a, b \in \mathbb{R}$. Construct an explicit bijection between $(0, 1)$ and $[a, b)$.

Prove or Disprove

F. Decide whether the statement is true or false. If true, prove it. If false, find a counterexample.

1) $\forall x \; \varnothing \subseteq x$.

2) $\varnothing = \{\varnothing\}$.

3) $A \cup (B \cap C) = (A \cup B) \cap C$.

4) $(A \setminus B) \setminus C = A \setminus (B \cup C)$.

5) $A = B$ if and only if $A \setminus B = B \setminus A$.

6) If $B \cap X = C \cap X$ for every set X, then $B = C$.

7) Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be defined by $f(x) = |x|$ and $A = \{x \in \mathbb{R} \mid x < 0\}$. Then $A \subseteq f^{-1}(\mathbb{R})$

8) Let $f : X \longrightarrow Y$ and $A \subseteq X$. Then $f(A^c) = [f(A)]^c$.

9) Let $f : X \longrightarrow Y$ and $B \subseteq Y$. Then $f^{-1}(B^c) = [f^{-1}(B)]^c$.

10) If there exists a bijection between X and Y, and there exists a bijection between Y and Z, there exists a bijection between X and Z.

11) The intersection of any two countable sets is countable.

Critical Thinking

G. Test your understanding.

1) In trying to prove that the product of any two odd numbers is also odd, a student showed that the product of 3 with 5 is odd. Why does this not work? What would be a better approach?

2) Note that for $f : X \longrightarrow Y$, $S_i \subseteq X$ for all $i \in I$, and $T_j \subseteq Y$ for all $j \in J$ equality fails in

$$f\left(\bigcap_{i\in I} S_i\right) \subseteq \bigcap_{i\in I} f(S_i)$$

but not

$$f^{-1}\left(\bigcap_{j\in J} T_j\right) = \bigcap_{j\in J} f^{-1}(T_j)$$

Can you explain why?

3) In an attempt to prove that a statement $S : P \implies Q$ is true for all natural numbers (i.e. that $S(n)$ holds for all $n \in \mathbb{N}$) a student started writing for the bases case: “Since $P(0)$ is false, statement $S(0)$ fails and therefore S is never true.” Correct this student’s reasoning.

4) For $f : X \longrightarrow Y$ and $S_1, S_2 \subseteq X$ we only have the containment

$$f(S_1 \setminus S_2) \subseteq f(S_1) \setminus f(S_2)$$

Find the conditions that make equality hold and prove your result.

5) We can identify $\mathbb{Q}$ with $\mathbb{Z} \times \mathbb{Z}^*$ in a natural way. Can this be used to define an explicit bijection between $\mathbb{Q}$ and $\mathbb{Z} \times \mathbb{Z}^*$? Why or why not?

6) Is it always true that for $f : X \longrightarrow Y$ and $A \subseteq X$ the restriction $f|_A : A \longrightarrow f(A)$ is a bijection?

7) Let $P_n = \{a_n x^n + \cdots a_1 x + a_0 \mid a_i \in \mathbb{Q}\}$ be the set of polynomials of degree at most n with rational coefficients. Is P_n a countable set?

8) How would you prove that there is no one-to-one function from $\mathbb{R}$ to $\mathbb{N}$?

Counterexamples and Constructions

H. Construct the desired example or counterexample.

1) Find an example that proves the statement “All two-digit numbers ending in 7 are prime” is false.[14]

[14] If your example starts with 5 you had the same idea as Grothendieck, according to legend.

2) Let $A_n = (-n, n) \subseteq \mathbb{R}$ for each $n \in \mathbb{N}$. Construct sets B_n so that $\bigcup_{i=1}^{n} A_n = \bigcup_{i=1}^{n} B_n$ but for any $i \neq j$ $B_i \cap B_j = \varnothing$.
3) Construct an example to show that in general for a function $f : X \longrightarrow Y$ and $S \subseteq X$, $S \not\supseteq f^{-1}(f(S))$.
4) Construct an example to show that in general for a function $f : X \longrightarrow Y$ and $T \subseteq Y$, $S \not\subseteq f(f^{-1}(T))$.
5) Prove that left inverse and right inverse functions may not be unique.
6) Find a bijection from $\mathbb{N} \times \mathbb{N}$ to $\mathbb{N}$.
7) Define a function $f : \mathbb{R} \longrightarrow \mathbb{R}$ and set A such that $|f(A)| = |\mathbb{N}|$.
8) Define a function $f : \mathbb{R} \longrightarrow \mathbb{R}$ such that $f|_{2\mathbb{Z}}(x) = x^2$ and $f|_{2\mathbb{Z}+1} = -x^2$.

Standard Problems

I. Solve.

1) Prove that if $A \subseteq B$ then $B^c \subseteq A^c$.
2) Prove that $A \subseteq B$ if and only if $2^A \subseteq 2^B$.
3) Prove that $A \cap B = \varnothing$ if and only if $A \subseteq B^c$.
4) Prove that $A \subseteq B$ if and only if $A \cup B = B$.
5) Prove that for any set A, $\bigcap_{B \in 2^A} B = \varnothing$.
6) Let $A_1 \subseteq A_2$ and $B_1 \subseteq B_2$. Prove that if $A_2 \cap B_2 = \varnothing$ then $A_1 \cap B_1 = \varnothing$.
7) Prove that $\{3n + 1 \mid n \in \mathbb{Z}\} = \{1 - 3m \mid m \in \mathbb{Z}\}$.
8) Let A be any finite subset of an ordered set X. Show that A has both a minimum element $m \in A$ such that $\forall a \in A$ $m \leq a$ and a maximum element $M \in A$ such that $\forall a \in A$ $M \geq a$.
9) Show that if $f : X \longrightarrow Y$, and $A, B \subseteq X$ are such that $A \subseteq B$ then $f(A) \subseteq f(B)$.
10) Let $f : X \longrightarrow Y$ be any function, let $S, S_1, S_2, S_i \subseteq X$ for all $i \in I$. Prove that if f is one-to-one, then
    i) $f\left(\bigcap_{i \in I} S_i\right) \subseteq \bigcap_{i \in I} f(S_i)$

ii) $f(S_1 \setminus S_2) = f(S_1) \setminus f(S_2)$
iii) $f^{-1}(f(S)) = S$

11) Let $f : X \longrightarrow Y$ be any function, let $T \subseteq Y$. Prove that if f is onto, then
$$f(f^{-1}(T)) = T$$

12) Let $f : X \longrightarrow Y$ be any function, let $S, S_1, S_2, S_i \subseteq X$ for all $i \in I$, and $T, T_1, T_2, T_j \subseteq Y$ for all $j \in J$. Prove that
   i) $f\left(\bigcup_{i\in I} S_i\right) = \bigcup_{i\in I} f(S_i)$
   ii) $f^{-1}\left(\bigcup_{j\in J} T_j\right) = \bigcup_{j\in J} f^{-1}(T_j)$
   iii) $f^{-1}\left(\bigcap_{j\in J} T_j\right) = \bigcap_{j\in J} f^{-1}(T_j)$
   iv) $f^{-1}(T_1 \setminus T_2) = f^{-1}(T_1) \setminus f^{-1}(T_2)$

13) Prove that a function $f : X \longrightarrow Y$ has a left inverse if and only if it is one-to-one.

14) Prove that a function $f : X \longrightarrow Y$ has a right inverse if and only if it is onto.

15) Prove that the function defined in this example is indeed a bijection.

16) Let X be a finite set. For each subset A of X, define a function $f_A : X \longrightarrow \{0, 1\}$ such that $\sum_{x\in X} f(x) = |A|$. Find an explicit bijection between 2^X and the set of functions from X to $\{0, 1\}$.

Bonus

J. Solve.

1) Show that if A_n is countable for each $n \in \mathbb{N}$, then $\bigcup_{n\in\mathbb{N}} A_n$ is countable.

2) Prove or disprove the following statement: $\forall x, y \in \mathbb{R} \setminus \mathbb{Q}\ x^y \in \mathbb{R} \setminus \mathbb{Q}$.

3) For each $n \in \mathbb{N}$ let $x_n \in \{0, 1\}^n = \underbrace{\{0, 1\} \times \cdots \times \{0, 1\}}_{n \text{ times}}$ be any n tuple of 0s an 1s. For a fixed n, construct an n-tuple y such that y differs from x_k in the kth entry.

4) Let $\mathbb{Q}(\sqrt{2})$ be the set $\{p + \sqrt{2}q \mid p, q \in \mathbb{Q}\}$. Prove that $|\mathbb{Q}(\sqrt{2})| = |\mathbb{Q}|$.

Chapter 2

Real Numbers

The *real* part of "real analysis" comes from the fact that we will be working primarily with the set of real numbers. To explain this choice we need to examine their properties. Some features of the real number line, like the fact that there exists a rational between any two real numbers, or that π and e are real numbers are often taken for granted. You may have known them to be true without examining them carefully.

We opt here for a definition of $\mathbb{R}$, rather than an explicit construction, for simplicity. The full list of axioms about the real number line is avoided here, under the assumption that the reader is familiar with the general notions of a field and an ordered set (both of which describe the set of rational numbers). These alone fail to highlight what makes the real number line special (i.e. why there isn't a "Rational Analysis" course to be studying for instead).

2.1 Real Numbers as an Ordered Field

To begin with, our definition should ensure the containment $\mathbb{Q} \subseteq \mathbb{R}$. In general an extension of a number set may not share all of its properties (consider $\mathbb{C}$, which is not ordered, as an extension of $\mathbb{R}$), but our starting point will be to at least have the algebraic structure and order of the rational numbers. $\mathbb{Q}$ is a *field* (a set where division and subtraction make sense) that is *ordered* (a set where we can compare elements with a relation like $\leq$). It can be proved that defining the set as an ordered field is enough to guarantee that it will include rational numbers, so this is already a good start.

Fact 2.1: Ordered fields contain $\mathbb{Q}$

Let $\mathbb{F}$ be an ordered field. Then there exists a map $\varphi : \mathbb{Q} \longrightarrow \mathbb{F}$ that respects all field operations[1].

Remark. This result does not exactly mean that $\mathbb{Q} \subseteq \mathbb{F}$, but rather that up to some relabeling we will find a $\mathbb{Q}$-like structure in $\mathbb{F}$.

What are the desirable properties of ordered fields? For one, the ability to be only a short distance away from any given number. Informally, when we think of operations that can make a number "small" we think of division so that the resulting number is "closer" to zero. This is not something that can be always done in arbitrary ordered fields, but works with rationals and real numbers. It is known as the Archimedean[2] property (or Archimedean principle), which comes in two formulations. In one, enough instances of something small can add up to exceed something big. In the other, numbers can be arbitrarily close to zero. Together they imply the existence of numbers that are infinitely large and infinitely small.

Definition: Archimedean Property

An ordered field $\mathbb{F}$ is said to have the *Archimedean property* if for any $x, y \in \mathbb{F}$, with $x > 0$, there exists $n \in \mathbb{N}$ such that

$$nx > y$$

If we let $y = 1$ and relabel x as ϵ, this reads as

$$\frac{1}{n} < \epsilon$$

We don't think of considering $\mathbb{N}$ or $\mathbb{Z}$ as Archimedean because, while they are ordered sets, they are not fields (i.e. division is not in general allowed). Similarly, we can't consider finite fields like $\mathbb{F}_2$ or $\mathbb{F}_9$ because finite fields are not ordered. The smallest field we can prove the result for is $\mathbb{Q}$ and, as a superset of $\mathbb{Q}$, $\mathbb{R}$ inherits the property.

[1] Such a map is called a *homomorphism* in abstract algebra.
[2] Pronounced `aar·kuh·mee·dee·uhn`.

Proposition 2.2: $\mathbb{Q}$ is Archimedean

Let $p, q \in \mathbb{Q}$ with $p > 0$. Then there exists $N \in \mathbb{N}$ such that $Np > q$.

Proof.
If $q \leq 0$ the result immediately follows, as $np > 0$ for any $n \in \mathbb{N}^*$. Suppose that $q > 0$ is of the form $q = \frac{n}{m}$ where n and m are relatively prime.[3] Similarly, we can assume $p = \frac{k}{l}$. Then from the fact that they share no common factors and both rational numbers are nonzero, we have $m \geq 1$ and $k \geq 1$, so $q < mq = n$ and $kp = l$. We can combine the inequalities to state

$$q < qm = n \leq nk = nlp$$

Setting $N = nl$ yields the desired result. □

Note that not all ordered fields have the Archimedean property[4] Being Archimedean is a stronger condition (i.e. a more difficult property to satisfy). A useful result that holds in *all* ordered fields is Bernoulli's inequality, which we present here without proof (its proof can be done by induction as an exercise).

Fact 2.3: Bernoulli's Inequality

Let $\mathbb{F}$ be an ordered field, $x \geq -1$ and $n \in \mathbb{N}$. Then

$$(1 + x)^n \geq 1 + nx$$

Bernoulli's inequality can be read as a result on function growth comparing polynomial functions of degree n with linear functions with leading coefficient n. However, it can also be used to show that exponents can be used to make numbers arbitrarily large or small, as we demonstrate in the following example.

Example 2.1.1:

Let $\epsilon > 0$ and $x < 1$ be elements of an ordered field with the Archimedean property. Then there exists a natural number $n \in \mathbb{N}$ such that

$$x^n < \epsilon$$

[3] I.e. they share no common factors other than 1.

[4] To learn more about those, look up non-Archimedean fields. Notable examples include the field of hyperreals, the Dehn field, and the Levi-Civita field.

Bernoulli's inequality guarantees the existence of y such that $x = 1 + y$, and we can use it again to show

$$x^m = (1+y)^m \geq 1 + my$$

for any $m \in \mathbb{N}$. By the Archimedean principle, we can find $n \in \mathbb{N}$ large enough so that $ny > \epsilon$, which yields

$$x^n \geq 1 + ny > 1 + \epsilon > \epsilon$$

Another result we will refer to often and holds true in ordered fields is the triangle inequality. However, it depends less on the field and more on the choice of the absolute value function, which we will later see defines a metric. This is to say, we can use the triangle inequality in a much wider variety of settings than that of ordered fields, but we present it here in order to facilitate the proofs in this chapter.

Proposition 2.4: Triangle Inequality

Let $\mathbb{F}$ be an ordered field and let $x, y \in \mathbb{F}$. Then

$$|x+y| \leq |x| + |y|$$

Proof.
Since substituting $-x$ for x and $-y$ for y makes no difference in the inequality, we may assume that $x + y \geq 0$. Then

$$|x+y| = x + y \leq |x| + |y|$$

□

Trick
The cases considered are for the sign of $x + y$, rather than x and y independently.

For easy reference, we include here some variations of the triangle inequality.

Fact 2.5: Triangle Inequality Variations

For any x, y, z in an ordered field $\mathbb{F}$ the following inequalities hold:

$$|x-z| \leq |x-y| + |x-z|$$
$$|x+y| \leq |x| + |y|$$

$$||x| - |y|| \leq |x - y|$$

What, then, makes $\mathbb{R}$ so special? It's not algebraic closure: remember $x^2 + 1 = 0$ has no solution in $\mathbb{R}$, and the algebraic closure of $\mathbb{Q}$ still fails to include transcendental numbers like e and π. It's not the ability to get arbitrarily close to other numbers (though this *is* important and it's why we mentioned the Archimedean property). It's that any value we can get close to is *also* a real number.

2.2 The Least Upper Bound Property

While informally this is described as saying that there are no gaps in the real number line, this is better stated as saying that "exact" bounds exist in $\mathbb{R}$. Consider $S = \{q \in \mathbb{Q} \mid q^2 < 2\}$. We can construct rational numbers that are in this set and are arbitrarily close to $\sqrt{2}$. We can see S is bounded above in $\mathbb{R}$ by all values y such that $y^2 \geq 2$, like for example $y = 5$. The least of these bounds will be exactly $\sqrt{2}$ but this number is not rational, so in $\mathbb{Q}$ this "exact" bound does not exist. More formally, these are called least upper bounds, which we define below.

Definition: Upper Bound, Supremum

A subset $S \subseteq \mathbb{R}$ is said to be bounded above if there exists $M \in \mathbb{R}$ such that for any $s \in S$ we have $s \leq M$. In this case we say M is an *upper bound* for S. Among all possible bounds, an upper bound α such that for any other upper bound M of S we have $\alpha \leq M$ is called the *least upper bound*.
An ordered field $\mathbb{F}$ where any nonempty set S bounded above has a least upper bound $\alpha \in \mathbb{F}$ is said to have the *least upper bound property*.

Note that bounds must be real numbers and ∞ and $-\infty$ are not in $\mathbb{R}$. With some changes, it is possible to find the supremum of *any* set, even if suprema aren't of much interest when the set is empty. While it may seem counterintuitive, it is vacuously true of any upper bound M of $\varnothing$ that $-\infty > M$.

Example 2.2.1:
The set $S = \{-x^2 + 2 \mid x \in \mathbb{R}\}$ is bounded above, as $-x^2 + 2 \leq 2$ for any $x \in \mathbb{R}$. In this case $\alpha = 2$ is the supremum of S. Any other number exceeding this value, such as $M = 4$ is simply an upper bound of S. Some sets, like $\mathbb{N} \subseteq \mathbb{R}$, are not bounded above.

Analogous to the concept of upper bounds and suprema are lower bounds and infima.

Definition: Lower Bound, Infimum

A set $S' \subseteq \mathbb{R}$ is said to be bounded below if there exists $m \in \mathbb{R}$ such that for any $s \in S$ we have $m \leq s$. In this case we call m a *lower bound* of S'.
If S' is bounded below, A lower bound β of S' such that if m is any other lower bound of S' we have $\beta \geq m$ is called the *greatest lower bound* of S'.
An ordered field $\mathbb{F}$ where any nonempty set S bounded below has a greatest lower bound $\beta \in \mathbb{F}$ is said to have the *greatest lower bound property*.
A set S' is said to be *bounded* (above and below) if there exists $M > 0$ such that for all $s \in S'$ $|s| < M$.

Example 2.2.2:
The set $\{\frac{1}{x} \mid x > 0\} \subseteq \mathbb{R}$ is bounded below by any negative number (as $x > 0$ implies $\frac{1}{x} > 0$). The greatest lower bound of the set is 0. This set is not bounded, as it has no upper bound.

The existence of suprema and infima is not guaranteed in general, but for any ordered field $\mathbb{F}$, the least upper bound property is equivalent to the greatest lower bound property.

Fact 2.6

Let $\mathbb{F}$ be an ordered field. Then $\mathbb{F}$ has the least upper bound property if and only if it has the greatest lower bound property.

It is often useful to consider that because it is the *least* upper bound, any value less than $\sup(S)$ fails to be an upper bound. Similarly, any value greater than $\inf(S)$ fails to be a lower bound.

Proposition 2.7

Let $S \subseteq \mathbb{R}$ be bounded above, $\alpha = \sup(S)$, and let $\epsilon > 0$. Then there exists $s \in S$ such that $s > \alpha - \epsilon$.

Proof.
Let $S \subseteq \mathbb{R}$ be bounded above, $\alpha = \sup(S)$ and $\epsilon > 0$. $\alpha > \alpha - \epsilon$ implies $\alpha - \epsilon$ is not an upper bound for S, so there exists $s \in S$ such that $s > \alpha - \epsilon$. □

Technique
Contrapositive statement.

Another way to think about it is to say that if we move any closer to S from $\sup(S)$ or $\inf(S)$ we inevitably end up inside the set (instead of bounding it). We therefore present equivalent definitions of the supremum and infimum:

Fact 2.8

$M = \sup(S)$ if and only if $\forall \epsilon > 0\ \exists s \in S :\ M - \epsilon < s \leq M$. Similarly, $m = \inf(S')$ if and only if $\forall \epsilon > 0\ \exists s' \in S' :\ m \leq s < m + \epsilon$.

Example 2.2.3:
Section 2.2 gives us a tool to find the supremum and infimum of a set by comparing against its elements (rather than by comparing a value against other bounds). When the sets are intervals, the order properties of $\mathbb{R}$ make the result immediate. Take for example $S = (3, \pi)$. Then $\inf(S) = 3$, as $3 + \epsilon$ can either fall in S (or past S, meaning $3 + \epsilon > s$ for all $s \in S$). Similarly, $\sup(S) = \pi$.

Technique
Use an equivalent definition.

For bounded intervals, the supremum and infimum are always straightforward.

Fact 2.9: Infima and Suprema of Intervals

Let $a, b \in \mathbb{R}$, with $a < b$. Then

$$\inf(a, b) = \inf(a, b] = \inf[a, b) = \inf[a, b] = a$$

and

$$\sup(a,b) = \sup(a,b] = \sup[a,b) = \sup[a,b] = b$$

If $A \subseteq \mathbb{R}$ is finite,

$$\inf(A) = \min(A)$$

and

$$\sup(A) = \max(A)$$

The equivalence in **??** justifies the use of only one of the two properties to define completeness below.

Definition: Complete Ordered Field

An ordered field $\mathbb{F}$ is said to be *complete* if for any nonempty $S \subseteq \mathbb{F}$ bounded above there exists a least upper bound $\alpha \in \mathbb{F}$.
We define $\mathbb{R}$ as a complete ordered field.

Since $\mathbb{R}$ is defined as having this property (i.e. this is not something we prove but something we assume), completeness is effectively an axiom. It is called the *Completeness Axiom of* $\mathbb{R}$[5].

2.3 Subsets and Supersets of Real Numbers

It can be shown that, up to isomorphism, $\mathbb{R}$ is the only ordered field containing $\mathbb{N}$ that satisfies the least upper bound property. To show that $\mathbb{Q}$ is not the same as $\mathbb{R}$ (given it is also an ordered field containing $\mathbb{N}$), we need to show the least upper bound property fails.

The *well-ordering principle* states that every nonempty subset of $\mathbb{N}$ has a minimum element. Using this along with the Archimedean property that $\mathbb{R}$ inherits from $\mathbb{Q}$ we can prove that any real number lies between two integers.

[5]Sometimes called Dedekind completeness

Lemma 2.10

Let $x \in \mathbb{R}$, $x \geq 0$, then there exists $n \in \mathbb{N}$ such that

$$n - 1 \leq x < n$$

Proof.
Let $S = \{m \in \mathbb{N} \mid m > x\}$. Using the Archimedean principle, we can find m such that $m = m \cdot 1 > x$, so $S \neq \varnothing$. Then by the well-ordering principle S has a minimum element, n. By construction we have $n > x$, and by minimality of n we have $n - 1 \leq x$, so the result holds. □

Advice
We can't prove the result by using $\lfloor x \rfloor$ and $\lceil x \rceil$: the argument would be circular.

As a corollary, we can conclude there is a rational number between any two real numbers. Though we will postpone the discussion of density until next chapter, here an intuition of being able to find rational numbers in between any two real numbers will suffice.

Corollary 2.11: $\mathbb{Q}$ is Dense in $\mathbb{R}$

Let $a, b \in \mathbb{R}$ with $b > a$. Then there exists $q \in \mathbb{Q} \cap (a, b)$.

Proof.
Using the fact that $b > a$ (and therefore $b - a \neq 0$ we deduce that $\frac{1}{b-a} \in \mathbb{R}$. We can use Archimedean property to find $n \in \mathbb{N}^*$ such that

$$n > \frac{1}{b - a}$$

which yields $nb - na > 1$. In particular, we have $nb > 1 + na$. Now, using lemma 2.10 we can find $m \in \mathbb{Z}$ such that

$$m \leq na < m + 1$$

Trick
If na and nb are more than one unit apart we can find integers between them.

Combining the two inequalities yields

$$na < m < nb$$

which is equivalent to

$$a < \frac{m}{n} < b$$

Then $q = \frac{m}{n} \in \mathbb{Q} \cap (a, b)$. □

Proposition 2.12: $\mathbb{Q}$ is not Complete

The set of rational numbers does not have the least upper bound property.

Technique

Proof by contradiction.

Proof.
Let $S = \{q \in \mathbb{Q} \mid q^2 < 2\}$. It is clear that S has upper bounds (for example, 2). Suppose for a contradiction that S has a *least* upper bound $\alpha \in \mathbb{Q}$. We will prove that in fact $\alpha = \sqrt{2} \notin \mathbb{Q}$.

If $\alpha < \sqrt{2}$, we could find $q \in (\alpha, \sqrt{2})$, say $q = \alpha + \epsilon$ for some $\epsilon > 0$. Then

$$\begin{aligned}
(q)^2 &= (\alpha+\epsilon)^2 \\
&= \alpha^2 + 2\underbrace{\alpha}_{<2}\epsilon + \underbrace{\epsilon^2}_{<\epsilon \text{ if } \epsilon<1} \\
&\leq \alpha^2 + 5\epsilon \\
&< 2 \\
\iff \epsilon &< \frac{2-\alpha^2}{5}
\end{aligned}$$

Since $\alpha \in \mathbb{Q}$, $\beta = \alpha + \frac{2-\alpha^2}{5} \in \mathbb{Q}$ and $\beta > \alpha$ while $\beta \in S$, contradicting our assumption that α is an upper bound of S.

If $\alpha > \sqrt{2}$, we could find $q \in (\sqrt{2}, \alpha)$, say $q = \alpha - \epsilon$. Then

$$\begin{aligned}
(q)^2 &= (\alpha-\epsilon)^2 \\
&= \alpha^2 - 2\underbrace{\alpha}_{\geq 2}\epsilon + \epsilon^2 \\
&> \alpha^2 - 4\epsilon \\
&> 2 \\
\iff \epsilon &< \frac{\alpha^2-2}{4}
\end{aligned}$$

We would then have $\beta = \alpha - \frac{\alpha^2-2}{4} \in \mathbb{Q}$ as an upper bound for S, since $\beta^2 < 2$, and $\beta < \alpha$, contradicting the fact that α should be the least upper bound.

It follows that $\alpha = \sqrt{2}$ and, as we showed earlier in proposition 1.3, $\sqrt{2} \notin \mathbb{Q}$, contrary to our initial assumption. We conclude that $\mathbb{Q}$ does not have the least upper bound property. ☐

All complete ordered fields are, in effect, equal to $\mathbb{R}$ and as we have just shown with the previous example, the set of real numbers is different from that of rational numbers. Completeness is a property worth defining a new number system for. Why? If we oversimplify a bit, let us state that anything we can approximate or get close to should exist so we can define things like limits. Though rationals exist arbitrarily close to $\sqrt{2}$, or π, these don't exist in the field and we therefore cannot define them as limits of approximating sequences using only $\mathbb{Q}$ as an underlying set. Completeness guarantees that limits exist even if we cannot explicitly say what they are, and this kind of existence result can be very powerful.

Where does that leave sequences like $x_n = 2^n$ which diverge by growing arbitrarily large? It can be convenient to consider that they also converge to infinity, but since the limit is not a real number we have to make a couple of amendments.

Definition: Extended Real Line

The *extended real line* consists of the real number system together with the symbols ∞ and $-\infty$:

$$\overline{\mathbb{R}} = \mathbb{R} \cup \{\infty\} \cup \{-\infty\}$$

A word on notation: in algebra, the notation $\overline{\mathbb{F}}$ is used to denote the algebraic closure of a field $\mathbb{F}$ and the extended real line is not the algebraic closure of $\mathbb{R}$ ($\mathbb{C}$ is). In some texts, the notation used is $\mathbb{R}^\#$, though in this writer's opinion the use of $\overline{\mathbb{R}}$ is more widespread. Alternatively, one may write $[-\infty, \infty]$ or $\mathbb{R} \cup \{-\infty, +\infty\}$.

In $\overline{\mathbb{R}}$ we can define the supremum of any subset of $\mathbb{R}$, rather than only bounded ones.

Definition: Supremum, Infimum

We define the *supremum* of an arbitrary set S, denoted by $\sup(S)$ as

$$\sup(S) = \begin{cases} \alpha & \text{if } S \text{ is bounded above with least upper bound } \alpha \\ \infty & \text{if } S \text{ is not bounded above} \\ -\infty & \text{if } S = \varnothing \end{cases}$$

We define the *infimum* of S', denoted by $\inf(S')$ as

$$\inf(S') = \begin{cases} \beta & \text{if } S \text{ is bounded below} \\ -\infty & \text{if } S \text{ is not bounded below} \\ \infty & \text{if } S = \varnothing \end{cases}$$

were β is the greatest lower bound of S'.

With this definition, the extended real number line has the least upper bound property and we can find the infima and suprema of arbitrary sets, rather than only bounded ones. Though it's a technical point, it will be necessary when defining limits.

2.4 Sequences

Approximating values through sets and inequalities like we did in the last example can be cumbersome. It is easier to have explicit values we can define and prove "get close to" the number we are interested, and from this point on that will be handled by sequences.

Definition: Sequence

Let X be a set. A *sequence* is a map $f : \mathbb{N} \longrightarrow X$. Each *term* of the sequence $f(n)$ is denoted by x_n, and the sequence itself (or all the relevant information in $f(\mathbb{N})$) is denoted by $(x_n)_{n\in\mathbb{N}}$ (sometimes $\{x_n\}_{n=1}^{\infty}$).

Though we are defining them here with domain $\mathbb{N}$, the index set for a sequence only has two requirements: it should be countable and ordered. We could therefore define sequences indexed with even numbers, odd numbers, or primes.

Advice

Build a mental library of examples/counterexamples.

Example 2.4.1: Some Types of Sequences

We list here some common sequence types. Consider these a part of your mental sequence library. Much like your mental function library, which may include polynomials, exponential functions, and trig functions, a mental sequence library is a good place to start looking for examples and counterexamples. We have

- constant: $x_n = a$ for some $a \in \mathbb{R}$.

$$(x_n)_{n\in\mathbb{N}} = (a, a, a, \ldots)$$

- harmonic: $x_n = \frac{1}{n}$ for each $n \in \mathbb{N}^*$.

$$(x_n)_{n\in\mathbb{N}} = \left(1, \frac{1}{2}, \frac{1}{3}, \ldots\right)$$

- alternating: $x_n = (-1)^n$.

$$(x_n)_{n\in\mathbb{N}} = (1, -1, 1, \ldots)$$

Sequences can either be defined explicitly (with a formula for the nth term) or recursively (where a term depends on those preceding it). When using a formula to describe $(x_n)_{n\in\mathbb{N}}$, it is possible to define a function $f : \mathbb{R} \longrightarrow \mathbb{R}$ such that $f|_{\mathbb{N}}(\mathbb{R}) = x_n$.

Example 2.4.2: Sequences as Functions

Sequences can be seen as discrete subsets of the image of real-valued functions (which may not match the formula for the nth term). Take for example the alternating sequence given by $x_n = (-1)^n$. As a function $f : \mathbb{N} \longrightarrow \mathbb{R}$ this is a good definition but if we change the domain from $\mathbb{N}$ to $\mathbb{R}$, the function is no longer well defined. For instance, evaluating at $\frac{1}{2}$ would not yield a real number. We can, however, see this sequence as the restriction of $\cos : \mathbb{R} \longrightarrow [-1, 1]$, to the set $S = \{x \in \mathbb{R} \mid \exists n \in \mathbb{N}\ x = \pi n\}$. Note, however, that this is not the only possible extension of f to $\mathbb{R}$, not even the only continuous extension. In general, discrete functions allow for simpler rules than their continuous counterparts, and in this way sequences offer an advantage.

Other than through explicit formulas, we can also define a sequence *recursively*, where one terms depends on the previous one(s). In the case where one term only depends on the one preceding it, we can write this formally as

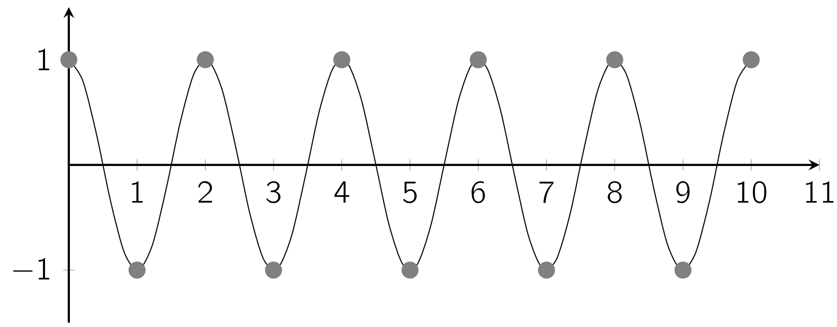

Figure 2.1: The alternating sequence $(-1)^n$ as part of $\cos : \mathbb{R} \longrightarrow [-1, 1]$.

the sequence given by $x_0 = a_0$, $x_{n+1} = f(x_n)$ for some function $f : \mathbb{R} \longrightarrow \mathbb{R}$. In particular, we could set $x_0 = 1$, $x_{n+1} = n + 1 + x_n$ results in

$$(0, 1, 3, 6, 10, \ldots)$$

Common examples of sequences described recursively include arithmetic sequences ($a_n = a_{n-1} + d$), geometric sequences ($a_n = a_{n-1} \cdot r$), the Fibonacci sequence ($a_{n+1} = a_n + a_{n-1}$), and $n!$ ($a_n = n \cdots a_{n-1}$).

The examples listed above cover a variety of different "end behaviors": as n increases, the sequence may approach a specific real number (like $\frac{1}{n}$), go back and forth between values (like $(-1)^n$) or continue to grow indefinitely (like $n!$). We can describe these outcomes in terms of convergence and divergence.

Definition: Bounded, Convergent, Divergent

Let $X = \mathbb{Q}$ or $X = \mathbb{R}$[a]. We say that a sequence $(x_n)_{n\in\mathbb{N}}$ is *bounded* if there exists $M > 0$ such that for all $n \in \mathbb{N}$ we have $|x_n| < M$. We say that the sequence is *convergent* if for some $x \in \mathbb{R}$ and any $\epsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that if $n \geq n_0$ then $|x_n - x| < \epsilon$. In this case we say that x is the *limit* of the sequence and may write

$$\lim_{n\to\infty} x_n = x$$

or $x_n \longrightarrow x$ for short. A sequence is *divergent* if it does not converge.

[a] We will define more general sequences in future chapters.

Example 2.4.3: Bounded and Unbounded Sequences

The sequence defined by $x_n = n^2$ is unbounded. Here it is easy to see that for any candidate bound M, there exists some $n \in \mathbb{N}$ such that $n^2 > M$. Since it is not bounded, the sequence cannot converge and must be divergent.

This is not to say that all bounded sequences are convergent, as the alternating sequence $(-1)^n$ is clearly bounded but cannot be said to converge to anything. The only good candidates for a limit are either -1 or 1 (Why?). If we choose 1 as our limit and set $\epsilon = 1$, for any $n_0 \in \mathbb{N}$ and $n \geq n_0$ there will be infinitely many terms where $x_n = -1$ and $|x_n - 1| > 1$, contradicting the fact that 1 is the limit.

Example 2.4.4: Convergent Sequence

On the other hand, $(y_n)_{n\in\mathbb{N}}$ defined by $y_n = \frac{n^2+n}{n^2}$ is bounded. For example, if we let $M = 2$ we can easily see that $|2n^2| \geq |n^2 + n|$ so that $M > y_n$ for any $n \geq 1$. This is not the *best* bound, it is simply *a* bound and that is enough. In this case we can prove that the sequence indeed converges to 1. To find the limit we can either compute it directly (as a quotient of polynomials of equal degrees.[6]

Seen as a quotient of polynomials, we have

Technique

Limit of a quotient of polynomials.

$$\lim_{n\to\infty} \frac{n^2 + n}{n^2} = \lim_{n\to\infty} \frac{1}{1} = 1$$

Alternatively, using L'Hôpital's rule[7] yields

Technique

Limit of a quotient of functions with L'Hôpital's rule.

$$\lim_{n\to\infty} \frac{n^2 + n}{n^2} = \lim_{n\to\infty} \frac{2n + 1}{2n} = \lim_{n\to\infty} \frac{2}{2} = 1$$

To prove that this is indeed the limit, using the definition above, we start with an arbitrary ϵ and compute the distance:

$$\begin{aligned} \left|\frac{n^2 + 2n}{n^2} - 1\right| &= \left|\frac{n^2 + 2n - n^2}{n^2}\right| \\ &= \left|\frac{2n}{n^2}\right| \end{aligned}$$

[6]If $f(x)$ and $g(x)$ are polynomials of degree n with leading coefficients a_n and b_n, $\lim x \to \infty \frac{f(x)}{g(x)} = \frac{a_n}{b_n}$

[7]If $f(x), g(x) \to \infty$ when $x \to \infty$, then $\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{f'(x)}{g'(x)}$

$$= \left|\frac{2}{n}\right|$$

Using the Archimedean property, we can find $n \in \mathbb{N}$ such that $\frac{2}{n} < \epsilon$, which proves that $(y_n)_{n\in\mathbb{N}}$ converges to 1 as desired.

If we can break down a sequence into a combination of others that are known to converge, we can deduce its limit. We list here some useful results to that effect.

Fact 2.13: Sequence Limits

Let $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ be sequences in $\mathbb{R}$ such that $x_n \longrightarrow x$ and $y_n \longrightarrow y$. Then

- $\lim_{n\to\infty}(x_n + y_n) = x + y = \lim_{n\to\infty} x_n + \lim_{n\to\infty} y_n$
- $\lim_{n\to\infty} x_n y_n = xy = (\lim_{n\to\infty} x_n)(\lim_{n\to\infty} y_n)$
- $\lim_{n\to\infty} cx_n = cx = c\lim_{n\to\infty} x_n$ for any real number c
- $\lim_{n\to\infty} \frac{x_n}{y_n} = \frac{x}{y} = \frac{\lim_{n\to\infty} x_n}{\lim_{n\to\infty} y_n}$ provided $y_n \neq 0$ and $y \neq 0$.

We can use these properties to find the limits of more complex expressions.

Example 2.4.5:

Since the limit of a sequence, if it exists, has to be a real number, unbounded sequences are always divergent. Equivalently, we can say that every convergent sequence is bounded. It can be proved, as follows, that if a sequence converges the limit is unique, so that the limit of a sequence is always well defined. The techniques to find limits are not one size fits all, so we encourage the reader to consider the ones presented here but remember that these are by no means the only ones available (or even "the best" for any given problem, which may be a matter of personal taste).

Fact 2.14

Let $(x_n)_{n\in\mathbb{N}}$ be a real-valued sequence. If $x_n \longrightarrow y$ and $x_n \longrightarrow z$ for $y, z \in \mathbb{R}$, then $y = z$.

Example 2.4.6: Finding the Limit of a Recursive Sequence

When a sequence is defined recursively and it converges, we can use the fact that

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} x_{n+1}$$

to find the limit.

Let $a > 0$ and consider $(x_n)_{n\in\mathbb{N}}$ where

$$x_0 = a + 1 \text{ and } x_{n+1} = x_n \left(1 - \frac{x_n^2 - a}{2x_n^2}\right)$$

If $L = \lim_{n\to\infty} x_n$, as n gets arbitrarily large we have

$$\begin{aligned} L &= L\left(1 - \frac{L^2 - a}{2L^2}\right) \\ L\left[1 - \left(1 - \frac{L^2 - a}{2L^2}\right)\right] &= 0 \\ L\left(\frac{L^2 - a}{2L^2}\right) &= 0 \\ \frac{L^2 - a}{2L} &= 0 \end{aligned}$$

Boundedness is a useful feature (after all, unbounded sequences can't converge) but is not enough to determine convergence or limits. In short, boundedness only provides "static" bounds: a horizontal band of uniform thickness where all terms of the sequence can be found. Better than that is a "dynamic bound," one that changes as we move further along in the terms of the sequence. For example, we can bound one sequence with another: the terms $\frac{\sin(n)}{n}$ are hard to list or find patterns in, but are bounded above by $\frac{1}{n}$ and below by $\frac{-1}{n}$. Since both bounding sequences converge to zero, it follows that $\frac{\sin(n)}{n}$ must also converge to zero. This result is summarized in the following theorem:

Theorem 2.15: Squeeze Theorem

Let $(x_n)_{n\in\mathbb{N}}$, $(y_n)_{n\in\mathbb{N}}$ and $(z_n)_{n\in\mathbb{N}}$ be real-valued sequences such that $x_n \leq y_n \leq z_n$ for all $n \in \mathbb{N}$ and suppose that

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} z_n = y$$

for some $y \in \mathbb{R}$. Then

$$\lim_{n\to\infty} y_n = y$$

Proof.
Let $\epsilon > 0$. By the definition of convergence, there must exist natural numbers n_1 and n_2 such that for all $n \geq n_1$ $|x_n - y| < \epsilon$ and for all $n \geq n_2$ $|z_n - y| < \epsilon$. If we let $n_0 = \max\{n_1, n_2\}$ then for all $n \geq n_0$ we have that both $|x_n - y| < \epsilon$ and $|z_n - y| < \epsilon$. This, along with $x_n \leq y_n \leq z_n$ implies

Trick
Taking the maximum guarantees this value will work for both sequences.

$$-\epsilon < x_n - y < y_n - y < z_n - y < \epsilon$$

which yields $|y_n - y| < \epsilon$. Thus

$$\lim_{n\to\infty} y_n = y$$

□

Since rational numbers are dense in $\mathbb{R}$, for any real number x we can construct a sequence in $\mathbb{Q}$ that converges to x. The proof of this result relies on the Squeeze Theorem, as can be seen below.

Theorem 2.16

Let $x \in \mathbb{R}$. Then there exists a sequence $(q_n)_{n\in\mathbb{N}}$ of rational numbers such that $q_n \longrightarrow x$.

Proof.
Let $\epsilon > 0$. For each $n \in \mathbb{N}$, we can use corollary 2.11 to find $q_n \in \left(x - \frac{1}{n}, x + \frac{1}{n}\right)$. By construction we have $x - \frac{1}{n} \leq q_n \leq x + \frac{1}{n}$, and since both the sequences $x - \frac{1}{n}$ and $x + \frac{1}{n}$ converge to x □

What we are looking for is not exactly the supremum or infimum of the set of sequence terms either.

The usual criterion for convergence of an arbitrary sequence $(x_n)_{n\in\mathbb{N}}$ to a limit L relies on the distance between x_n and L becoming infinitesimally small. Cauchy sequences display a stronger form of convergence that only depends on the terms of the sequence, and not the value they approach.

Definition: Cauchy Sequence

A *Cauchy sequence* is a sequence $(x_n)_{n\in\mathbb{N}}$ such that for every $\epsilon > 0$ there exists $n_0 \in \mathbb{N}$ so that for any $n, m \in \mathbb{N}$ $n > n_0$, $m > n_0$ we have $|x_m - x_n| < \epsilon$.

To say that $\mathbb{R}$ is complete is to say that it is a metric space containing the limit of every Cauchy sequence.

2.5 Series

Practice

Memory

A. Recall definitions to complete the sentences.

1) The supremum of $\varnothing$ should be greater than every other upper bound for $\varnothing$. Why does it make sense to define $\sup(\varnothing) = -\infty$.

Computation

1) s

Quick Applications

B. Use definitions to prove the desired result.

1) Let S be a nonempty bounded subset of $\mathbb{R}$ and let $a > 0$. Prove that

$$\sup(aS) = a\sup(S)$$

and

$$\inf(S) = a \inf(S)$$

Focus on Technique

C. Use the techniques demonstrated throughout the chapter to prove the following results.

1) Let $x > 1$ and $\alpha > 0$ be elements of an Archimedean ordered field. Show that there exists $n \in \mathbb{N}$ such that

$$x^n > \alpha$$

2) Use Bernoulli's inequality to prove that

   i) if $x > 1$ and $\epsilon > 0$ then there exists $n \in \mathbb{N}$ such that $x^n > \epsilon$.

   ii) if $y < 1$ and $\epsilon > 0$ then there exists $n \in \mathbb{N}$ such that $y^n < \epsilon$.

3) Use the AM-GM Theorem to prove Bernoulli's inequality.

4) The *well-ordering principle* states that every nonempty subset of $\mathbb{N}$ has a minimum element. Use this principle to prove that if $x \in \mathbb{R}$, $x \geq 0$, then there exists $n \in \mathbb{N}$ such that $n \leq x < n + 1$. This proves lemma 2.10.

5) Prove that the set of irrational numbers is dense in $\mathbb{R}$.

6) Prove the results stated in **??**

Prove or Disprove

D. Decide whether the statement is true or false. If true, prove it. If false, find a counterexample.

1) If $1 = \sum_{n \in \mathbb{N}} c_n$ and $0 \leq c_n$ for each n then there is a $d < 1$ such that $c_n < d^n$ for each n.

1) Prove that if $\{a_n\}_{n \in \mathbb{N}}$ is a sequence of real numbers then

$$\liminf_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right| \leq \frac{1}{\limsup_{n\to\infty} |a_n|^{1/n}} \leq \frac{1}{\liminf_{n\to\infty} |a_n|^{1/n}} \leq \limsup_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right|$$

Critical Thinking

E. Test your understanding.

1) Is $\overline{\mathbb{R}}$ a complete ordered field?

Counterexamples and Constructions

F. Construct an example or counterexample with the desired property.

1) Prove that if $a, b \in \mathbb{R}$ and $a < b$ there exists $q \in \mathbb{Q} \cap (a, b)$.
2) Give an example of a series which is absolutely convergent but not uniformly convergent.
3) A well-known theorem states that if a sequence $\{f_n\}_{n\in\mathbb{N}}$ of continuous functions converges uniformly to a function f on E, then f is continuous. Give examples of sequences $\{f_n\}_{n\in\mathbb{N}}$ of functions on $[a, b]$ such that
   i) $\sum_{n\in\mathbb{N}} f_n(x)$ converges absolutely for all x but does not converge uniformly
   ii) $\sum_{n\in\mathbb{N}} f_n(x)$ converges uniformly, but does not converge absolutely for any x

   i) Give an example to show that the converse is not true; that is, a sequence of continuous functions may converge to a continuous function, although the convergence is not uniform.
   ii) State conditions under which we may be able to assert the converse.

Standard Problems

G. s

1) Find the limits:
   i)

$$\lim_{n\to\infty} \sqrt[n]{n^2}$$

ii)
$$\lim_{n\to\infty} \frac{n^{1000}}{(1.0001)^n}$$

iii)
$$\lim_{n\to\infty} (0,9999)^n$$

iv)
$$\lim_{n\to\infty} \sum_{k=2}^{n} \frac{1}{k \log k}$$

2) (3) Suppose $\sum_{n\in\mathbb{N}} na_n$ is a convergent series of positive, nonincreasing terms. Prove
$$\lim_{n\to\infty} a_n = 0$$

3) Suppose $\lim_{n\to\infty} n^p u_n = A$. Prove that $\sum_{n\in\mathbb{N}} u_n$ converges for $p > 1$ and $A < \infty$ and diverges otherwise.

4) If $\{s_n\}_{n\in\mathbb{N}}$ is a sequence of real numbers and
$$b_n = \frac{s_1 + s_2 + \cdots + s_n}{n}$$
for each n, prove
$$\limsup_{n\to\infty} b_n \leq \limsup_{n\to\infty} s_n$$

5) If $\{a_n\}_{n\in\mathbb{N}}$is a sequence of real numbers and
$$b_n = \frac{a_1 + a_2 + \cdots + a_n}{n}$$
for each n, prove that $a_n \longrightarrow a$ implies $b_n \longrightarrow a$. **Nathan says:** *An interesting addendum here, if you want to add it; you can actually regularize divergent sequences/series using this summation. For example, the series with general term $(-1)^k$ obviously diverges, but its Cesaro averages converge to 1/2.*

6) Assume $\{a_n\}_{n\in\mathbb{N}}$ is a sequence of positive numbers such that $a_n \longrightarrow a$. Show that
$$(a_1 a_2 \cdots a_n)^{1/n} \longrightarrow a$$

7) A sequence $\{a_n\}$ is called *subadditive* if $a_{n+m} \leq a_n + a_m$, for any $n, m \geq 1$. Show that, if a_n is a subadditive sequence, then $\{\frac{a_n}{n}\}$ converges, and its limit is equal to $\inf_{n\geq 1} \frac{a_n}{n}$.

Solution.

Set $L := \inf_{n\geq 1} \frac{a_n}{n}$, and fix $\epsilon > 0$. By properties of infimum, there is an $N \geq 1$ such that $a_N < N(L+\epsilon)$. Let A be the maximum of the first N elements of $\{a_n\}$. Now, for any $n > N$, we can write $n = pN + r$, for some $p \in \mathbb{N}$, by the division algorithm. Making use of the subadditivity of $\{a_n\}$, we find that

$$a_n = a_{pN+r} \leq pa_N + a_r \leq pa_n + A.$$

Therefore, we obtain

$$\frac{a_n}{n} \leq \frac{pa_N + A}{n} \leq \frac{pN(L+\epsilon)}{n} + \frac{A}{n} \to L + \epsilon$$

as $n \to \infty$, as a consequence of $\frac{pN}{n}$ tending to 1 as $n \to \infty$. Since ϵ was arbitrary, the theorem follows. □

[8]

8) Prove that the series

$$\sum_{n\in\mathbb{N}^*} (-1)^n \frac{x^2+n}{n^2}$$

converges uniformly on every interval $[-a, a]$ but does not converge absolutely for any value of x.

9) Prove

   i) If $\sum_{n\in\mathbb{N}} |a_n| < \infty$ then $\sum_{n\in\mathbb{N}} |a_n|^2 < \infty$.

   ii) If $a_n \geq 0$ and $\sum_{n\in\mathbb{N}} a_n < \infty$ then $\sum_{n\in\mathbb{N}} \frac{\sqrt{a_n}}{n} < \infty$

10) (2) Show that if $\{a_n\}_{n\in\mathbb{N}}$ is a positive sequence and $\sum_{n\in\mathbb{N}} a_n^2 < \infty$ then

$$\sum_{n\in\mathbb{N}^*} \frac{a_n}{n^\alpha} < \infty$$

for every $\alpha > 1/2$.

[8]This is called Fekete's lemma, and has some interesting combinatorial applications. For example, one can show that the number of self avoiding random walks on a given lattice of length n, ℓ_n, is log-subadditive (the log of the sequence is subadditive), and so the limit $\mu = \lim_{n\to\infty} \ell_n^{1/n}$ exists, and is called the connective constant of the lattice. - Nathan

11) Let $\sum_{n\in\mathbb{N}} a_n$ be a series of positive terms and let k be a fixed positive integer. Prove that if

$$\limsup_{n\to\infty} \frac{a_n + k}{a_n} < 1$$

then $\sum_{n\in\mathbb{N}} a_n$ converges while if

$$\liminf_{n\to\infty} \frac{a_n + k}{a_n} > 1$$

then $\sum_{n\in\mathbb{N}} a_n$ diverges.

12) (2) Suppose $s_n = \sum_{k\leq n} a_k$ is bounded and $\{b_k\}_{k\in\mathbb{N}}$ decreases to zero. Prove that $\sum_{n\in\mathbb{N}} a_k b_k$ converges.

13) If $\sum_{n\in\mathbb{N}} c_n\xi_n$ converges for every sequence $\{\xi_n\}_{n\in\mathbb{N}}$ such that $\xi_n \longrightarrow 0$ as $n \to \infty$, prove that $\sum_{n\in\mathbb{N}} c_n$ converges absolutely.

14) Show that if $\{f_n\}_{n\in\mathbb{N}}$ is a monotone increasing sequence of continuous real functions defined on $[0, 1]$ that converges pointwise to a continuous function f, then the convergence is uniform.

15) (2) Let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence of continuous functions which converges uniformly to a function f on a set E. Prove that

$$\lim_{n\to\infty} f_n(x_n) = f(x)$$

for every sequence of points $\{x_n\}_{n\in\mathbb{N}}$ in E such that $x_n \longrightarrow x \in E$. Is the converse true?

16) Establish the convergence or divergence of the following:

i)
$$\sum_{n\in\mathbb{N}^*} \frac{(-1)^{n-1}n}{n^2+1}$$

ii)
$$\sum_{n\geq 2} \frac{1}{n\ln^2 n}$$

iii)
$$\sum_{n\in\mathbb{N}^*} \frac{(-1)^{n-1}2^n}{n^2}$$

17) Suppose $\sum_{n\in\mathbb{N}} a_n = \infty$, where $a_n > 0$ for each n. Establish the conditions on $\{a_n\}_{n\in\mathbb{N}}$ that determine convergence or divergence of the following series, providing examples when necessary.

i)
$$\sum_{n\in\mathbb{N}} a_n^2$$

ii)
$$\sum_{n\in\mathbb{N}} \frac{a_n}{1+a_n}$$

iii)
$$\sum_{n\in\mathbb{N}} \log\left(\frac{a_{n+1}}{a_n}\right)$$

18) Suppose $a_n > 0$, $s_n = a_1 + a_2 + \cdots + a_n$, and $\sum_{n\in\mathbb{N}^*} a_n$ diverges. What can be said about the convergence or divergence of $\sum_{n\in\mathbb{N}} \frac{a_n}{1+na_n}$ and $\sum_{n\in\mathbb{N}} \frac{a_n}{1+n^2 a_n}$?

19) (2) Let $a_n > 0$ for each $n \in \mathbb{N}$. Show that if $\sum_{n\in\mathbb{N}} a_n$ converges then $\sum_{n\in\mathbb{N}} a_n^2$ and $\sum_{n\in\mathbb{N}} a_n a_{n+1}$ also converge. Does the result still hold if we no longer require $a_n > 0$?

20) Given that $\sum_{n\in\mathbb{N}} a_n$ converges, where each $a_n > 0$, prove that $\sum_{n\in\mathbb{N}} (a_n a_{n+1})^{1/2}$ also converges.

21) Suppose $a_n > 0$ and $\sum_{n\in\mathbb{N}} a_n$ converges. Let $r_n = \sum k \geq na_k$. Prove

i)
$$\sum_{n\in\mathbb{N}} \frac{a_n}{r_n}$$

diverges

ii)
$$\sum_{n\in\mathbb{N}} \frac{a_n}{\sqrt{r_n}}$$

converges.

22) Let $f(x) = \sum_{n\in\mathbb{N}} a_n x^n$. Prove that $f(x)$ and $f'(x)$ have the same radius of convergence.

23) (2) Let
$$f(x) = \begin{cases} e^{-\frac{1}{x^2}} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

i) Show that $f^{(n)}(0)$ exists for all $n \geq 1$.

ii) Show that the Taylor series about 0 generated by f converges everywhere on $\mathbb{R}$ but that it represents f only at the origin.

24) Find the Taylor series for the function

$$f(x) = \frac{e^x}{1+x^2}$$

and compute its radius of convergence.

25) (2) Consider

$$f(x) = \sum_{n\in\mathbb{N}} \frac{1}{1+n^2x}$$

i) For what values of x does the series converge absolutely?

ii) On what intervals does it converge uniformly?

iii) On what intervals does it fail to converge uniformly?

iv) Is f continuous wherever the series converges?

v) Is f bounded?

vi) Does $\{f_n'(x)\}_{n\in\mathbb{N}}$ converge to $f'(x)$ for all x?

26) Does the sequence defined by the following functions converge uniformly?In which intervals?

$$f_n(x) = \frac{x}{n}e^{-\frac{x}{n}}$$

$x \in [0, \infty)$

27) Prove that $\sum_{n\in\mathbb{N}^*} \frac{1}{p_n}$, where p_n is the nth prime, converges.

28) Discuss the convergence/divergence of

$$\sum_{n\in\mathbb{N}^*} \left(1 + \frac{1}{2} + \cdots + \frac{1}{n}\right) \frac{\sin(nx)}{n}$$

29) Show that $\sum_{n\in\mathbb{N}^*} \frac{\sin(nx)}{n}$ is convergent for all real numbers x.

30) Prove that the series $\sum_{n\in\mathbb{N}^*} \frac{\sin(nx)}{n}$ is not uniformly convergent on any interval that includes the origin.

31) Determine the values of p that make the following series convergent:

$$\sum_{n\geq 3} \frac{1}{n \log n(\log(\log n))^p}$$

32) Prove that $\int_{\pi}^{\infty} \frac{\sin x}{x} dx$ converges conditionally.

33) Show that the sequence $\{f_n\}_{n\in\mathbb{N}^*}$ defined by $f_n(x) = \sin(\ln(n+x))$ has a uniformly convergent subsequence on $[1, 100]$.

34) Does the sequence $\{f_n\}_{n\in\mathbb{N}^*}$ with $f_n(x) = \sin(\ln(nx))$ have a uniformly convergent subsequence on $[1, 2]$? Why?

35) Assume that $\{F_n\}_{n\in\mathbb{N}^*}$ converges uniformly to the continuous function F on $\mathbb{R}$. Show that

$$F_n\left(x + \frac{1}{x}\right) \longrightarrow F(x)$$

for all $x \in \mathbb{R}$.

36) IF $\{f_n\}_{n\in\mathbb{N}}$ is a sequence of functions such that for each n $f_n'(x)$ exists for all $x \in [a, b]$, is continuous for all $x \in (a, b)$, $f_n \longrightarrow f$ on $[a, b]$ and $f_n' \overset{\text{unif}}{\longrightarrow} g$ on $[a, b]$, prove $g = f'$ on $[a, b]$.

37) Let $x_{n+1} \cos(x_n)$, $x_0 = 1$. Show that $\{x_n\}_{n\in\mathbb{N}}$ is a convergent sequence.

38) Suppose $f : \mathbb{R} \longrightarrow \mathbb{R}$ and f' are continuous over $\mathbb{R}$. For $x_0 \in [z-1, z+1]$, define the sequence $x_{n+1} = f(x_n)$. Consider the limit $\lim_{n\to\infty} f(x_n)$. Describe its behavior in the cases when

   i) $|f'(y)| < 1$ for all $y \in [z-1, z+1]$
   ii) $|f'(y)| \geq 1$ for all $y \in [z-1, z+1]$

and prove a convergence result.

39) (2) Prove that if $\sum_{n\in\mathbb{N}} a_n$ converges to A, $s_n = \sum_{k\leq n} a_n$ and $\{b_n\}_{n\in\mathbb{N}}$ is defined by

$$b_n = \frac{s_1 + s_2 + \cdots + s_n}{n}$$

for each n then $b_n \longrightarrow A$.

40) Prove there is no continuous function f on $\mathbb{R}$ such that

   - $f(x)$ is rational if x is irrational, and
   - $f(x)$ is irrational if x is rational

41) Let $f : [0, 2] \longrightarrow \mathbb{R}$ be defined as

$$f(x) = \begin{cases} 0 & x \in [0, 1) \\ 2 & x \in [1, 2] \end{cases}$$

Does there exist a function $g : [0, 2] \longrightarrow \mathbb{R}$ such that $g'(x) = f(x)$?

42) Let M be an upper bound for $S \subset \mathbb{R}$. Prove that $M = \sup S$ if and only if for every $\epsilon > 0$ there is an $s \in S$ such that $M - \epsilon < s \leq M$. Formulate a similar statement regarding the $\inf S$.

Solution.

$(\Longrightarrow)$
Let $M = \sup S$ and $\epsilon > 0$. Suppose for a contradiction that there are no elements of S in the interval $(M - \epsilon, M]$. Then $M - \epsilon$ would be an upper bound for S, and since $\epsilon > 0$ it would be a lower bound than M, contradicting its minimality. Therefore there must exist an element $s \in S$ such that $s \in (M - \epsilon, M]$ (i.e. $M - \epsilon < s \leq M$).
$(\Longleftarrow)$
We prove the contrapositive statement. Suppose that $M \neq \sup S$, that is it is not the least upper bound for S. Then there must exist another bound $B < M$ for S (i.e. $s \leq B\ \forall s \in S$.) Since B is an upper bound, no element of S is above it, and given it is different from M, $M - B > 0$. If we set $\epsilon = M - B$, then no element of S lies in $(M - \epsilon, M] = (B, M]$ (i.e. for some $\epsilon > 0$ no $s \in S$ satisfies $M - \epsilon < s \leq M$).

A parallel statement about the infimum is:

Let m be a lower bound for $S \subset \mathbb{R}$. Then $m = \inf S$ if and only if for every $\epsilon > 0$ there is an $s \in S$ such that $M + \epsilon > s \geq M$. □

43) Let $\{x_n\}$ be a sequence of real numbers and put

$$\overline{s}_{x,N} = \sup\{x_N, x_{N+1}, x_{N+2}, \ldots\} \quad \text{and} \quad \underline{s}_{x,N} = \inf\{x_N, x_{N+1}, x_{N+2}, \ldots\}$$

where $N = 1, 2, \ldots$. Define

$$\overline{\lim}\, x_n = \inf_{N \geq 1} \overline{s}_{x,N} = \lim_{N \to \infty} \overline{s}_{x,N} \quad \text{and} \quad \underline{\lim}\, x_n = \sup_{N \geq 1} \underline{s}_{x,N} = \lim_{N \to \infty} \underline{s}_{x,N}$$

Prove that

$$\overline{\lim}\, x_n + \underline{\lim}\, y_n \leq \overline{\lim}\,(x_n + y_n) \leq \overline{\lim}\, x_n + \overline{\lim}\, y_n$$

provided that the left and right sides are not of the form $\infty - \infty$.

Bonus

1) Prove that the series

$$\left(\frac{3}{2}-\frac{4}{3}\right)+\left(\frac{5}{4}-\frac{6}{5}\right)+\cdots\left(\frac{2n+1}{2n}-\frac{2n+2}{2n+1}\right)+\cdots$$

is convergent, where the series

$$\frac{3}{2}-\frac{4}{3}+\frac{5}{4}-\frac{6}{5}+\cdots\frac{2n+1}{2n}-\frac{2n+2}{2n+1}+\cdots$$

is divergent.

Additional Resources

For further reading of the construction by Dedekind cuts (which may be of logical and historical interest), see the appendix to Chapter 1 in [Rud76] (for a brief version) or Chapter IV of [Lan66] (for an extended version).

Index

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